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METHODS OF ANALYSIS FOR ANISOTROPIC MEDIA

A THESIS

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Stephen William Toland

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To My Parents

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LIST OF SYMBOLS

$\underline{\underline{S}}$	Stress tensor
$\underline{\underline{E}}$	Strain tensor
$\underline{u}$	Displacement tensor
$\underline{\underline{I}}$	Idemfactor (basic isotropic dyadic)
$\underset{\sim}{J}$	Basic fourth order isotropic tensor
U	Strain energy function
f	Volume concentration factor
E	Modulus of elasticity
μ	Shear modulus
ν	Poisson's ratio
$\varkappa$	Bulk modulus
$\underline{r}$	Position vector
$\underset{\sim}{L}$	Fourth order stiffness tensor
$\underset{\sim}{M}$	Fourth order compliance tensor
$C_{ijk\ell}$	Scalar components of $\underset{\sim}{L}$
σ	Scalar stress components
ϵ	Scalar strain components
$\underline{\underline{U}}$	Somigliana displacement dyadic
φ	Newtonian potential
ψ	Biharmonic potential
B	Bounding surface
D	Region bounded by B
V	Volume of D
ϑ	Volumetric dilatation

SUMMARY

The purpose of this work is to investigate analytical methods used in the study of aelotropic media. The work is comprised of a detailed description of the techniques currently being used by other analysts. An attempt has been made to unify the notation used, tensors are used almost exclusively, and to show relations between the various methods. Tensor notation was used because of its conciseness of form and presentation, and because of its invariant nature involving transformation of coordinates. It is the author's hope that other students will acquire an interest in the field of composite materials by reading this presentation.

The various methods that are investigated in this work are the use of energy formulations to predict bounds on the bulk material properties of a composite, the use of the Navier field equation to formulate a boundary value problem from which exact results concerning bulk material properties may be obtained, the use of a "self-consistent" technique to improve upon exact results, and the use of potential theory for the analysis of the inclusion problem. The fact that most of the investigation to date have been concerned with the prediction of over-all bulk material properties is pointed out. This is due largely to the fact that the problem of analyzing the internal stress, strain, and displacement fields of a composite material is extremely difficult.

CHAPTER 0

INTRODUCTION

The purpose of this research is to investigate the analytical methods presently being used in the study of composite materials. The methods to be studied include the use of energy formulations for the predictions of overall bulk material properties, the use of potential theory for the analysis of the inclusion problem, and stress versus strain relations for the exact analysis of bulk material properties. The advantages and disadvantages of the various methods will be discussed and interrelations between different approaches will be pointed out.

Analysis of mechanical behavior of aelotropic materials is important for several reasons. Two of the principal reasons are: (1) many materials of importance to today's advancing technology are heterogeneous (e.g., reinforced rubbers and plastics, concrete, multiphase alloys, and ceramic compounds); (2) today's technological advances have required that designers use materials with specified properties. Hence, the ability to predict material properties of aelotropic media is a prerequisite for designing new materials which can satisfy modern demands.

The following terminology will be employed throughout the investigation: An aelotropic media will also be referred to as being heterogeneous, anisotropic, or a composite. As a composite material it will be composed of various constituent phase materials. Those composites which consist of N number of phases will be called "multiphase" media. An important special case of the above is the "two-phase" media usually composed of a base material known as the matrix and a second phase with different material properties which is distributed in the matrix in the form of inclusions. The inclusions may or may not be of a specified size and shape, and their distribution may be random or form a regular array. At phase interfaces the usual continuity conditions on stress vectors and displacements are always specified. Therefore, it is assumed that no local discontinuities in the form of holes or separations are present between the constituent phases. This condition implies perfect adhesion between the inclusions and the matrix.

Of the various methods used to investigate heterogeneous media, the most commonly used is that of the theory of bulk mechanical behavior. By bulk mechanical behavior is meant the over-all or macroscopic behavior of the media and the values of the macroscopic elastic material moduli. The theory of actual field analysis in heterogeneous media is much less developed since the problem of determining the actual internal stress and strain fields of the media is very complex. Actually, the advances in bulk behavior stem from attempts to bypass the much harder problem of field determination.

Survey of Literature

The first attempt to investigate an aelotropic media was Einstein's analysis (1) of the viscosity of a dilute suspension of rigid spheres in a Newtonian viscous fluid, in 1906. The theory of dilute suspensions was further investigated in 1923 by Jeffery (2) who analyzed Einstein's problem for rigid ellipsoidal particles, and in 1932 by Taylor (3) who used viscous spheres with surface tension for inclusions. Since then dilute suspensions have been analyzed for many different types of inclusions in a viscous fluid with factors such as slippage and friction along the interfaces considered.

The problem treated in the present investigation, that of elastic inclusions dispersed in a matrix having different elastic moduli (the elastic-elastic case), was first treated by Bruggeman (4) in 1937. He found the correct expression for the bulk modulus. It was Dewey (5) in 1947 who first correctly analyzed the problem for the shear modulus. However, this work seems to have been overlooked by most of the later investigators. Without knowledge of Dewey's work the special case of spherical voids in the elastic matrix was solved by Mackenzie (6) in 1951, and the case of rigid spheres for inclusions by Hashin (7) in 1956. All of the above mentioned works dealt with what are called dilute suspensions; e.g., very small volume concentrations of inclusions.

The initial investigations of aelotropic media consisting of finite volume concentrations of inclusions also were studies of particles suspended in a viscous fluid. The first work done with this type of media composed of two elastic materials was an approximation of the overall elastic properties conducted by Kerner (8) in 1956. Since then, several advances have been made due to work done by Krivolaz and Cherevko (9) in 1959 in which inclusions consisting of elastic spheres was treated by a perturbation method; Hill and Crossley (10) in 1963 who analyzed an elastic matrix reinforced by an equally oriented array of elastic fibers of equal square cross sections; and Hashin and Rosen (11) who studied equally oriented fibers of hollow or solid circular cross sections, in 1964.

The most often cited work in the area of determination of the internal stress and strain fields is Eshelby's (12) investigations of the elastic field of an ellipsoidal inclusions embedded in an infinite matrix, done in 1957. The results of this work are used frequently by other investigators of related problems. The important aspects of the results are discussed in a later section of the present work. In 1961, Eshelby (13) published results of further investigations of elastic inclusions in which he also reviewed some of the more recent methods devised for studying anisotropic crystals.

A number of works are to be found concerning the predictions of bounds on the macroscopic material properties. Most of these treatments incorporate energy theorems and variational techniques from the theory of elasticity. The central problem of bounding techniques is treated by Sokolnikoff (14). Some of the references of this approach are Paul (15) in 1960, Hashin and Shtrickman (16, 17) in 1962 and 1963, and Hill (10) in 1963. The works of Hashin, Shtrickman, and Hill are improvements on Paul's work, and treat the problem by use of what is known as the elastic polarization tensor for description of the elastic fields. This idea was first introduced by Eshelby (12). Some of the above treatments were for true multiphase media.

Recently the bulk problem has been investigated by incorporating what is called the "smearing out" or "self-consistent" technique. Hill (18), 1965, analyzes two phase composites with ellipsoidal elastic inclusions using the "self-consistent" theory, and Budiansky (19), 1965, uses Eshelby's results to analyze multiphase media by the "self-consistent" approach.

The literature on the subject of composite materials has been very comprehensively reviewed by Z. Hashin. This survey appears as "Theory of Mechanical Behavior of Heterogeneous Media" in _Applied Mechanics Reviews 17_, p. 1 (1964). Most of the above survey was taken from Hashin's report.

CHAPTER I

BOUNDS ON THE ELASTIC CONSTANTS OF A POLYPHASE MEDIA

With the growing demands of industry on the specification of the mechanical properties of the materials required to fill the needs of advancing technology and construction, the designer is faced with the problem of attempting to create hybrid, so to speak, materials. The word "hybrid" is used to denote those materials which are not found in nature. Thus, the need arises for the development and use of polyphase materials. The use of these types of materials presents the designer with the problem of predicting what combinations of materials will yield the sought after mechanical properties.

As long as the materials are isotropic the designer may prescribe the mechanical properties for several basic structural elements; such as beams, plates, shells, etc. as long as the usual linear strength of materials or known elasticity relations between the strain distribution and stress-strain relations are applicable to the particular problem. Some of these types of relations are $\sigma = Mc/I$, $\sigma = P/A$, $\Delta = PL/AE$, series solutions for plates, and simple shells of revolution formulas. However, if the designer is forced to use a polyphase material, which is anisotropic, he must resort to the theory of elasticity for answers. The theory of elasticity, however, presents equations for stress versus strain which are in general too complex to lend themselves to direct solutions or results. In other words, the theory of elasticity will supply equations and relations for the variables involved in the formulation of the problem to be solved; nevertheless, this by no means implies that the analyst will be capable of finding any form; explicit, implicit, or otherwise; of a closed solution to the problem.

Obviously, the need arises to try to develop a method whereby the designer may at least determine some relationship between the known properties of the constituent materials and the required properties of the polyphase media to within some preassigned degree of accuracy. Hence, establish bounds on the composite material's properties by using the known data available; i.e., the properties of the separate constituents and their volume percentage in the total composite. One approach to such a relation would be to assume that there exists an "equivalent" isotropic material which exhibits elastic constants (E^*, μ^*). E^* and μ^* are such that the hypothetical isotropic material behaves exactly as the anisotropic material is required to behave when subjected to some impressed boundary tractions and/or displacements. Thus, if the anisotropic specimen was subjected to a simple uniaxial tension test along some x axis, the average E_x determined or exhibited across an arbitrary cross section of the specimen should be equal to $E_x{}^*$ of the "equivalent" isotropic material.

Analysis

The composite material will be presumed to consist of two phases; phase one will be referred to as the matrix or base material and phase two will consist of particles dispersed in the matrix. It will be assumed that the composite media is uniform and isotropic in the large (macroscopically) with effective Young's modulus, Poisson's ratio, and shear modulus denoted by E^*, ν^*, μ^* respectively. Furthermore, it will be assumed that in the large effective uniform stress, $\underline{\underline{S}}^*$, and uniform strain, $\underline{\underline{E}}^*$, distributions exist which satisfy the equilibrium equations, boundary conditions, and compatibility relations. Although the above distributions don't actually exist uniformly in the composite material, the assumptions are made in order to imply that macroscopically the anisotropic media behaves as an "equivalent" isotropic media. Figure 1 shows a cross-section of the composite material consisting of the matrix (phase one) and the particle inclusions (phase two).

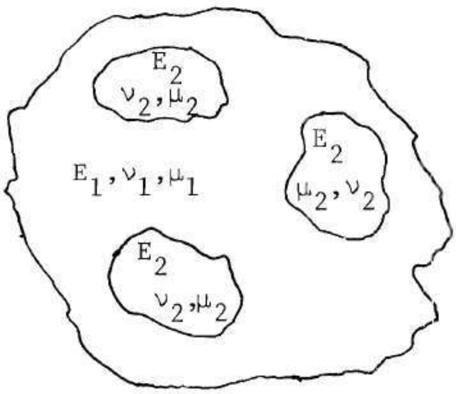

Figure 1.

Composite Cross-Section.

B. Paul (15) has shown that by use of two of the energy theorems of linear elasticity one may establish bounds on the assumed effective properties of an equivalent isotropic material. The strain energy U stored in the specimen is:

$$U = \frac{1}{2} \int_D \underline{\underline{S}}:\underline{\underline{E}} \, dv \tag{1-1}$$

where

$$\underline{\underline{S}} = \sigma_{ij} \, \hat{i}_i \hat{i}_j$$

$$\underline{\underline{E}} = \frac{1}{2}(\nabla \underline{u} + \underline{u}\nabla) = \varepsilon_{ij} \, \hat{i}_i \hat{i}_j$$

for the linearized case and dv denotes a differential volume element of the volume domain D. If the specimen is subjected to a simple uniaxial tension test; then, macroscopically the uniform stress field $\underline{\underline{S}}_x$ and the uniform strain field $\underline{\underline{E}}_x$ along the axis of the direction of tension may be measured. $\underline{\underline{S}}_x$ and $\underline{\underline{E}}_x$ are assumed distributed over a volume which includes a great many inclusions; and, although, in the neighborhood of an inclusion the local nonhomogeniety of the media disallows any truly uniform stress or strain fields, the average values of the normal stress and strain over a sufficiently large area must be equal to the macroscopically measured values $\underline{\underline{S}}_x$ and $\underline{\underline{E}}_x$. In the actual test the ratio of $\underline{\underline{S}}_x$ and $\underline{\underline{E}}_x$ is measured, and it is this ratio which defines E*, the equivalent elastic modulus of the media. Therefore

$$E^* = \underline{\underline{S}}_x : \underline{\underline{E}}_x^{-1} \quad . \tag{1-2}$$

However,

$$\underline{\underline{S}}_x = \sigma_x \, \hat{i}_1 \hat{i}_1 = (\hat{i}_1 \cdot \underline{\underline{S}}^* \cdot \hat{i}_1) \hat{i}_1 \hat{i}_1$$

$$\underline{\underline{E}}_x = \epsilon \, \hat{i}_1 \hat{i}_1 = (\hat{i}_1 \cdot \underline{\underline{E}}^* \cdot \hat{i}_1) \hat{i}_1 \hat{i}_1$$

$$\underline{\underline{E}}^* = \epsilon_{ij} \, \hat{i}_i \hat{i}_j = \text{total strain field}$$

$$\underline{\underline{S}}^* = \sigma_{ij} \, \hat{i}_i \hat{i}_j = \text{total stress field} \; .$$

So, one sees that (1-2) is the familiar $E^* = \frac{\sigma_x}{\epsilon}$. Thus, the strain energy may be written as

$$U = \tfrac{1}{2} \, \frac{\underline{\underline{S}}_x : \underline{\underline{S}}_x}{E^*} \; V \tag{1-3}$$

or

$$U = \tfrac{1}{2}\, E^*(\underline{\underline{E}}_x : \underline{\underline{E}}_x)V \qquad (1\text{-}3a)$$

since (1-2) may be expressed as $\underline{\underline{E}}_x E^* = \underline{\underline{S}}_x$.

The theorem of least work will provide a lower bound for E^*.

Theorem I

Let the tractions, $\underline{t}_n = \hat{n}\cdot\underline{\underline{S}}$, be completely specified over the surface B of a body, and let $\underline{\underline{S}}^o$ be a state of stress which satisfies the stress equations of equilibrium and the specified boundary conditions. Define U^o as the strain energy computed from the state $\underline{\underline{S}}^o$ by means of (1-1) and Hooke's law. Then, the actual strain energy U in the body due to $\underline{t}_n$ cannot exceed U^o; i.e., $U \le U^o$.

Hooke's law may be stated in the following form for an isotropic material:

$$\underline{\underline{S}} = \lambda\vartheta\, \underline{\underline{I}} + 2\mu\underline{\underline{E}} \qquad (1\text{-}4)$$

where λ and μ are Lame's constants and $\vartheta = \underline{\underline{E}}:\underline{\underline{I}}$ is the volumetric dilatation. In order to use Theorem I an admissible $\underline{\underline{S}}$ has to be assumed throughout the specimen. Such a stress field is $\underline{\underline{S}}_x$ as defined above. Therefore

$$U^o = \tfrac{1}{2}\int_D \frac{\underline{\underline{S}}_x : \underline{\underline{S}}_x}{E^*}\, dv = \frac{\underline{\underline{S}}_x : \underline{\underline{S}}_x}{2}\int_D \frac{1}{E^*}\, dv \ . \qquad (1\text{-}5)$$

Now, if one assumes a linear relationship between the effective compliance $\frac{1}{E^*}$ and the constituent compliances $\frac{1}{E_1}$, $\frac{1}{E_2}$ of the form

$$\frac{1}{E^*} = \frac{1-f}{E_1} + \frac{f}{E_2}$$

as proposed by MacDonald and Ransley (20), where f is the volume percentage of material two; then (1-5) may be written as

$$U^o = \frac{\underline{\underline{S}}_x : \underline{\underline{S}}_x}{2}\left[\frac{1-f}{E_1} + \frac{f}{E_2}\right]V \ . \tag{1-6}$$

Thus, by using equations (1-3a), (1-6), and the theorem of least work, one may obtain

$$\frac{1}{E^*} \leq \frac{1-f}{E_1} + \frac{f}{E_2} \tag{1-7}$$

or more conveniently

$$E^* \geq \frac{1}{\left(\frac{1-f}{E_1}\right) + \left(\frac{f}{E_2}\right)} \ . \tag{1-8}$$

Equation (1-8) gives a lower bound on E* and agrees with Paul's result.

An upper bound on E* may be obtained from the theorem of minimum potential energy.

Theorem II

Let the displacements $\underline{u}_B$ be completely specified over the surface B of a body (except where corresponding $\underline{t}_n$ vanish), and let $\underline{\underline{E}}^o$ be any compatible strain field which satisfies $\underline{u}_B$. Define U^o as the strain energy computed from the state $\underline{\underline{E}}^o$ by means of (1-1) and Hooke's Law. Then the actual strain energy U in the deformed body cannot exceed U^o; i.e., $U \leq U^o$.

In the tension test the specimen elongates by an amount $(E_x:I)L$. Thus, an admissible strain field for this $\underline{u}_B$ is

$$\epsilon_x = (\underline{\underline{E}}_x : \underline{\underline{I}}) = \epsilon \ ; \quad \epsilon_y = \epsilon_z = -\, m\epsilon \qquad (1\text{-}9)$$

$$\underline{\underline{E}}^* - \frac{1}{3}(\underline{\underline{E}}^* : \underline{\underline{I}})\underline{\underline{I}} = \underline{\underline{E}}^{*\prime} = 0$$

$\underline{\underline{E}}^{*\prime}$ represents the shear strains

$$\underline{\underline{E}}_x : \underline{\underline{I}} = \hat{i}_1 \cdot \underline{\underline{E}}^* \cdot \hat{i}_1 = \epsilon$$

where "m" is an unspecified constant. Thus, $\underline{\underline{E}}^*$ generates the following stresses

$$\hat{i}_1 \cdot \underline{\underline{S}}^* \cdot \hat{i}_1 = \sigma_x = \frac{(\underline{\underline{E}}_x : \underline{\underline{I}})E^*(1-\nu-2\nu m)}{(1-\nu-2\nu^2)} \qquad (1\text{-}10)$$

$$\hat{i}_2 \cdot \underline{\underline{S}}^* \cdot \hat{i}_2 = \sigma_y = \frac{(\underline{\underline{E}}_x : \underline{\underline{I}})E^*(\nu-m)}{(1-\nu-2\nu^2)}$$

$$\hat{i}_3 \cdot \underline{\underline{S}}^* \cdot \hat{i}_3 = \sigma_z = \sigma_y$$

$$\underline{\underline{S}}^* - \frac{1}{3}\, S^*_1 \ \underline{\underline{I}} = 0$$

where

$$S^*_1 = \underline{\underline{S}}^* : \underline{\underline{I}} \ .$$

Using equations (1-1), (1-9), and (1-10) the strain energy may be written as

$$U^{\circ} = \frac{\epsilon^2}{2} \int_D \left[\frac{1-\nu-4\nu m+2m^2}{1-\nu-2\nu^2}\right] E^* dv \; . \tag{1-11}$$

If one assumes a linear relationship for the stiffness similar to that assumed for the compliance

$$E^* = (1-f)E_1 + fE_2 \; . \tag{1-12}$$

Then equation (1-11) becomes

$$U^{\circ} = \frac{\epsilon^2}{2} V\left[\frac{(1-\nu_1-4\nu_1 m+2m^2)(1-f)E_1}{1-\nu_1-2\nu_1^2} + \frac{(1-\nu_2-4\nu_2 m+2m^2)fE_2}{1-\nu_2-2\nu_2^2}\right] . \tag{1-13}$$

Using equations (1-13), (1-3a), theorem II, and $\epsilon^2 = \underline{\underline{E}}_x : \underline{\underline{E}}_x$ one obtains

$$E^* \leq \frac{1-\nu_1+2m(m-2\nu_1)}{1-\nu_1-2\nu_1^2}(1-f)E_1 + \frac{1-\nu_2+2m(m-2\nu_2)}{1-\nu_2-2\nu_2^2} fE_2 \; . \tag{1-14}$$

Equation (1-14) is an upper bound for E* and agrees with Paul's result. Paul points out that although inequality (1-14) is valid for any choice of "m", the best results are obtained for U° being a minimum. Since $\nu \leq \frac{1}{2}$, it is easy to show that $\frac{\partial^2 U^{\circ}}{\partial m^2} > 0$; and, hence U° has a relative minimum for the values of m for which $U^{\circ}{}_{,m} = 0$. Moreover, this relative minimum is an absolute minimum since U° is a quadratic in m which exhibits positive values for very large values of positive or negative m. Thus, $U^{\circ}{}_{,m} = 0$ occurs at

$$m = \frac{\nu_1(1+\nu_2)(1-2\nu_2)(1-f)E_1 + \nu_2(1+\nu_1)(1-2\nu_1)fE_2}{(1+\nu_2)(1-2\nu_2)(1-f)E_1 + (1+\nu_1)(1-2\nu_1)fE_2} \; . \tag{1-15}$$

Since in the limiting cases where f approaches 0 or 1 causing m to approach ν_1 or ν_2 respectively, m may be thought of as an effective Poisson's ratio. Furthermore, for the special case where $\nu_1 = \nu_2 = \nu$ equation (1-15) gives $m = \nu$. This reduces (1-14) to

$$E^* \leq (1-f)E_1 + fE_2 \; . \tag{1-16}$$

The assumed relation for the compliance and stiffness of the composite material are simply the most readily available linear functional relationships. From equation (1-12) one easily sees that $E^* = E_1$ for $f = 0$ and $E^* = E_2$ for $f = 1$ which are the desired extremes.

Example

For illustrative purposes only, suppose that a two-phase composite material were composed of aluminum as the matrix and steel as the dispersed material. Then, one would have

$$E_1 = 10 \times 10 \text{ psi} \qquad E_2 = 30 \times 10 \text{ psi}$$

$$\mu_1 = 3.8 \times 10 \text{ psi} \qquad \mu_2 = 12 \times 10 \text{ psi}$$

$$\nu_1 = 0.33 \qquad \nu_2 = 0.25 \; .$$

Therefore, equation (1-8) could be written as

$$E^*_L = \frac{E_1 E_2}{E_2 + (E_1 - E_2)f} = \frac{300 \times 10^6}{(30 - 20f) \text{ psi}} \; .$$

Similarly, equation (1-15) yields

$$m_{min} = \frac{2.06 + 1.23f}{6.25 + 6.92f}$$

and equation (1-14) yields

$$E^*_u = \frac{0.667 + 2m(m - 0.667)}{0.444} (10^7)(1 - f)$$

$$+ \frac{0.75 + 2m(m - 0.5)}{0.625} (3 \times 10^7)(f) .$$

Hence, one may compute the values for E^*_L and E^*_u that are shown tabulated in Table I.

Figure II is a plot of the values of E^*_L and E^*_u versus f as given in Table I. As can be observed, the plot of E^*_u differs only slightly from the straight line that would result if in inequality (1-14) $\nu_1 = \nu_2$. The values plotted for E^*_u do deviate from the straight line; however, the deviation is so slight as to be imperceptible on the scale of Figure II. The smallness of the deviation would lead one to believe that the E^*_u as predicted by the rigorous method used, that of minimum potential energy, is essentially insensitive to the effects of Poisson's ratio. Furthermore, the expressions developed for E^*_L and E^*_u establish absolute bounds on E* with respect to f. In other words, if f is the only influencing factor on the values of E*, then expressions (1-8) and (1-14) can be expected to bound E* below and above respectively.

As can be seen from the example cited the difference between E^*_u and E^*_L may be rather large for $0.5 \leq f \geq 0.95$. For the example used with $f = 0.6$ a measure of the error introduced is

$$\text{Error} = \frac{2(E^*_u - E^*_L)}{E^*_u + E^*_L} = \frac{10.86}{38.77} (100) = 28\%$$

whereas without the benefit of the analysis

Table 1. Values of Bounds on E.

f	m	$E_u^* \times 10^{-6}$	$E_u^* \times 10^{-6}$
1.00	0.250	30.000	30.000
0.95	0.252	27.273	29.016
0.90	0.254	25.000	28.031
0.85	0.256	23.077	27.046
0.80	0.258	21.429	26.059
0.75	0.261	20.000	25.071
0.70	0.263	18.750	24.082
0.65	0.266	17.647	23.092
0.60	0.269	16.667	22.101
0.55	0.272	15.789	21.108
0.50	0.275	15.000	20.113
0.45	0.279	14.286	19.117
0.40	0.283	13.636	18.173
0.35	0.287	13.044	17.117
0.30	0.292	12.500	16.178
0.25	0.297	12.000	15.107
0.20	0.302	11.538	14.171
0.15	0.308	11.111	13.084
0.10	0.315	10.714	12.066
0.05	0.322	10.345	11.042
0.00	0.333	10.000	10.000

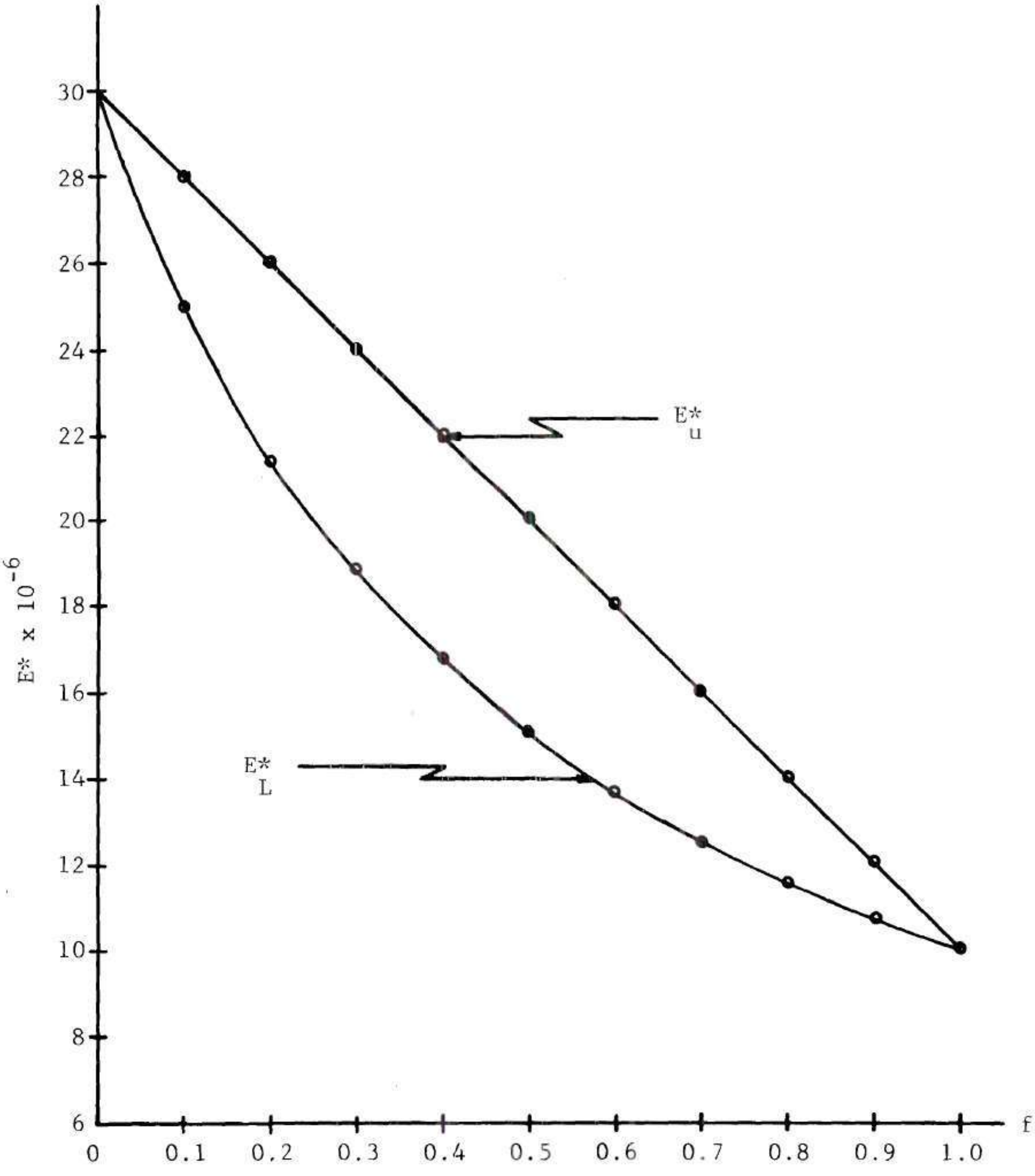

Figure 2. Paul's Method - E^*_u and E^*_L Versus f.

$$\text{Error} = \frac{2(E_2 - E_1)}{E_1 + E_2} = \frac{4}{4} \times 100 = 100\% \quad .$$

Thus, the use of energy theorems from elasticity to predict bounds on E* greatly reduce the error in the problem. Nevertheless, the error resulting from the analysis may be unacceptably large. One is therefore lead to believe that factors other than f influence E*.

The above analysis does not depend upon the size or shape of the particle inclusions or the way in which the inclusions are dispersed in the matrix. Generally, for composites which Hashin (21) calls finite suspensions, the value of E* will be dependent upon the size and shape of the particles, their orientations in the matrix with respect to the principal axes of the entire composite, and the interaction of their perturbation fields. A finite suspension is one in which the concentrations of the inclusions are no longer small. Also, the distance between inclusions is not great enough to negate their perturbation effects. Ways of dealing with this type of composite in an attempt to reduce the error introduced in the prediction of E* and other mechanical properties of the medium will be discussed in a later chapter.

Returning to the "equivalent" isotropic specimen, suppose that it is subjected to a pure shear $\underline{\underline{S}}_{\tau}$ which produces the macroscopically uniform shear strain $\underline{\underline{E}}_{\gamma}$. The following relations are apparent

$$\underline{\underline{S}}_{\tau} = \tau \hat{i}_1 \hat{i}_2 = (\hat{i}_1 \cdot \underline{\underline{S}}^* \cdot \hat{i}_2) \hat{i}_1 \hat{i}_2$$

$$\underline{\underline{S}}^* : \underline{\underline{I}} = S^*_1 = 0$$

$$\underline{\underline{E}}_{\gamma} = \epsilon \hat{i}_1 \hat{i}_2 = (\hat{i}_1 \cdot \underline{\underline{E}}^* \cdot \hat{i}_2) \hat{i}_1 \hat{i}_2$$

$$\underline{\underline{E}}^{*}:\underline{\underline{I}} = \vartheta^{*} = 0$$

where $\underline{\underline{S}}^{*}$ and $\underline{\underline{E}}^{*}$ represent, as before, the stress field and the strain field respectively in the specimen.

For such a state of stress

$$\mu^{*} = \underline{\underline{S}}_{\tau}:\underline{\underline{E}}_{\gamma}^{-1} \ . \tag{1-17}$$

Thus, the actual strain energy in the specimen may be expressed as follows using (1-1)

$$U = \frac{1}{2}\int_{D} \frac{\underline{\underline{S}}_{\tau}:\underline{\underline{S}}_{\tau}}{\mu^{*}}\, dv = \frac{\underline{\underline{S}}_{\tau}:\underline{\underline{S}}_{\tau}}{\mu^{*}}\, V \tag{1-18}$$

or

$$U = \frac{1}{2}\int_{D} \mu^{*}(\underline{\underline{E}}_{\gamma}:\underline{\underline{E}}_{\gamma})dv = (\tfrac{1}{2})\mu^{*}(\underline{\underline{E}}_{\gamma}:\underline{\underline{E}}_{\gamma})V \tag{1-18a}$$

Once again the theorem of least work will be used to provide a lower bound for μ^{*}. An admissible stress state which satisfies the requirements of theorem I is $\underline{\underline{S}}_{\tau}$. Thus,

$$U^{o} = \frac{1}{2}\int_{D} \frac{\underline{\underline{S}}_{\tau}:\underline{\underline{S}}_{\tau}}{\mu^{*}}\, dv = \frac{\underline{\underline{S}}_{\tau}:\underline{\underline{S}}_{\tau}}{2}\int_{D} \frac{1}{\mu^{*}}\, dv \ . \tag{1-19}$$

Assuming the simple volume weighting relation

$$\frac{1}{\mu^{*}} = \frac{1-f}{\mu_1} + \frac{f}{\mu_2} \tag{1-20}$$

one may write (1-19) as

$$U^{\circ} = \tfrac{1}{2}(\underline{\underline{S}}_{\tau} : \underline{\underline{S}}_{\tau}) \left(\frac{1-f}{\mu_1} + \frac{f}{\mu_2}\right)V \ . \tag{1-21}$$

Therefore, using theorem I, (1-18), and (1-21) one obtains

$$\frac{1}{\mu^*} \le \frac{1-f}{\mu_1} + \frac{f}{\mu_2} \tag{1-22}$$

or more conveniently

$$\mu^* \ge \frac{1}{\left(1 - \frac{f}{\mu_1}\right) + \left(\frac{f}{\mu_2}\right)} \ . \tag{1-23}$$

Equation (1-23) represents a lower bound on μ^*.

The theorem of minimum potential energy will be used to provide an upper bound for μ^*. An admissible strain field for $\underline{u}_B$ caused by the uniformly applied shear is $\underline{\underline{E}}_{\gamma}$. Therefore

$$\underline{\underline{E}}^* : \underline{\underline{I}} = \vartheta^* = 0 \tag{1-24}$$

$$\epsilon_{12} = \hat{i}_1 \cdot \underline{\underline{E}}^* \cdot \hat{i}_2 = \epsilon$$

$$\epsilon_{23} = \epsilon_{31} = 0 \ .$$

This strain field produces the following stresses

$$\underline{\underline{S}}^* : \underline{\underline{I}} = S^*_1 = 0 \tag{1-25}$$

$$(\hat{i}_1 \cdot \underline{\underline{S}}^* \cdot \hat{i}_2)\hat{i}_1\hat{i}_2 = \underline{\underline{S}}_{\tau} = \mu^* \underline{\underline{E}}_{\gamma}$$

$$(\hat{f}_2 \cdot \underline{\underline{S}}^* \cdot \hat{f}_3) = (\hat{f}_1 \cdot \underline{\underline{S}}^* \cdot \hat{f}_3) = 0$$

$$(\hat{f}_1 \cdot \underline{\underline{S}}^* \cdot \hat{f}_2) = \tau_{12} = 2\mu^* \varepsilon \ .$$

Equations (1-24) and (1-25) give

$$U^{\circ} = \int_D \mu^*(E_\gamma : E_\gamma) dv$$

$$U^{\circ} = (\tfrac{1}{2})(\underline{\underline{E}}_\gamma : \underline{\underline{E}}_\gamma) \int_D \mu^* dv \ . \qquad (1\text{-}26)$$

Assuming $\mu^* = (1-f)\mu_1 + f\mu_2$; theorem II, (1-18a), and (1-26) give

$$\mu^* \geq (1-f)\mu_1 + f\mu_2 \ . \qquad (1\text{-}27)$$

Equation (1-27) represents an upper bound on μ^*.

In the derivation of the upper bound for E^*, the following assumption was made

$$m = \nu^* \ . \qquad (1\text{-}28)$$

In the example demonstrating E^*_L and E^*_u values for m were tabulated. For $f = 0.55$, $m = 0.272$. The following well known relationship always hold

$$\nu^* = \frac{E^*}{2\mu^*} - 1 \ . \qquad (1\text{-}29)$$

For the same example and $f = 0.55$ equations (1-23) and (1-27) give

$$\mu^*_L = 6.09 \times 10^6 \text{ psi}$$

$$\mu_u^* = 8.31 \times 10^6 \text{ psi} .$$

Using (1-29) and the appropriate values of μ^* and E^*, one obtains

$$\nu_u^* = \frac{21.108}{12.176} - 1 = 0.73$$

$$\nu_L^* = \frac{15.79}{16.62} - 1 = - 0.05 .$$

These values represent extreme bounds for the value of ν^* with $f = 0.55$. Since E_u^* is an absolute upper bound for E^* and thus places no restrictions on the value of μ^* associated with it, the value of ν_u^* was computed using the smallest allowable value of μ^*; i.e., μ_u^*. Similar reasoning was used in computing ν_L^*. Nevertheless, perhaps a more logical assumption would be to associate μ_u^* with E_u^*. Thus, say that the material exhibits the value of μ^* that corresponds to the same relative value of E^*. With this in mind, (1-29) yields

$$\nu_{u'}^* = \frac{21.108}{16.62} = 1 = 0.269 . \qquad (1\text{-}30)$$

Thus, $\nu_{u'}^*$ differs from the calculated value $\nu^* = 0.272$ only by 0.003; a rather minute difference.

CHAPTER II

THE HOMOGENEOUS INCLUSION PROBLEM FOR BULK PROPERTIES

In order to reduce the error introduced by use of Paul's absolute bounds some sort of boundary value problem involving properties of the system other than the volume concentrations of the constituents may be studied. This problem in its most general form; i.e., random distribution of particles, m number of different constituents each contributing some n_i $(i = 1,\ldots,k)$ number of particles, non-uniformity of particle size and shape, and interaction of the perturbation effects of various particles; is extremely difficult even to formulate. A beginning approach is one that may be called the "homogeneous inclusion" problem for bulk properties.

Consider a composite material consisting of a matrix with bulk modulus and shear modulus denoted by $\varkappa_m$ and μ_m and particle inclusions with bulk and shear moduli denoted by $\varkappa_p$ and μ_p respectively. For this analysis $\varkappa$ and μ are taken as the independent isotropic elastic constants. Poisson's ratio ν and Young's modulus E are regarded as dependent properties. The following relations hold

$$\nu = \frac{\left(\frac{1}{2} - \frac{\mu}{3\varkappa}\right)}{\left(1 + \frac{\mu}{3\varkappa}\right)} \tag{2-1}$$

$$\frac{3}{E} = \frac{1}{\mu} + \frac{1}{3\varkappa}$$

$$E = 2\mu(1-\nu) = 3\kappa(1-2\nu) \;.$$

Again let f be the volume concentration of the particles in the matrix, and assume that f is small. By assuming small volume concentrations the following restrictions are implied.

i) Squares and higher powers of the volume concentrations f may be neglected in comparison with f itself.

ii) Between any two particles it is possible to find a region of neglible perturbation effects.

iii) At any point in the system perturbation effects of all particles are additive.

Assumption (ii) allows one to analyze the system by use of the "inclusion problem". For the "homogeneous inclusion" problem imagine an infinite elastic body, the matrix, which contains a single spherical inclusion of another elastic material. Thus, the perturbation effects of other inclusions may be neglected. Furthermore, continuity of the boundary between the matrix and the inclusion is assumed, and the matrix is subjected to some type of simple state of stress and strain. The assumption of a spherical inclusion is made for the sake of symmetry and simplicity.

The determination of the gross elastic constants of the dispersed system may be accomplished by use of the Navier field equation

$$(\lambda+\mu)\nabla\nabla\cdot \underline{u} + \mu \nabla^2 \underline{u} = 0 \;.$$

This field equation is used to determine the stress and displacement fields around the inclusion. The following boundary conditions are

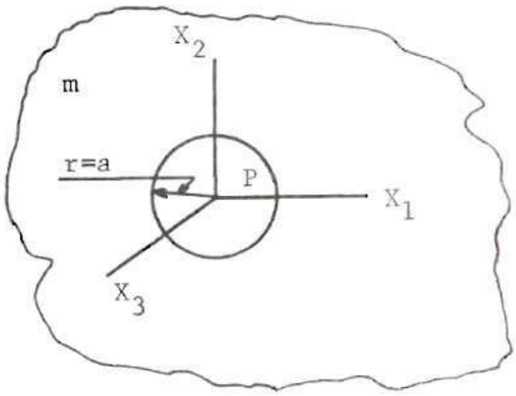

Figure 3.

A Single Inclusion.

prescribed:

a) $\underline{u} = \underline{u}_o$ @ $r \to \infty$

$$\left.\begin{array}{ll} \text{b)} & \underline{u}_s^{(p)} = \underline{u}_s^{(m)} \\ \text{c)} & \underline{\underline{S}}_s^{(p)} = \underline{\underline{S}}_s^{(m)} \end{array}\right\} \quad r = a$$

In the above λ is Lame's elastic modulus, $\nabla \cdot \underline{u} = \vartheta$ = volume dilatation, $\underline{u}$ = the displacement in Cartesian coordinates, $\underline{u}_s$ = the displacement in spherical coordinates, $\underline{\underline{S}}_s$ = the stress tensor in spherical coordinates, and a = the radius of the inclusion. One may assume that the boundary value problem has been solved and write the solution outside the inclusion as

$$\underline{u} = \underline{u}_o + \underline{u}' \tag{2-2}$$

$$\underline{\underline{S}} = \underline{\underline{S}}_{os} + \underline{\underline{S}}'_s \tag{2-3}$$

where the zero subscript refers to the field at infinity and the prime denotes the perturbation due to the particle.

Boundary condition (a) restricts $\underline{u}'$ and $\underline{\underline{S}}'_s$ to vanish at infinity. Furthermore, since only expressions for gross elastic properties of the system are of interest and not the effects of various types of solutions to (2-2) and (2-3), the simplifying restriction that $\underline{u}_o = \frac{1}{3}\, \vartheta\, \underline{\underline{I}} \cdot \underline{x}$ may be imposed. This restriction implies that the zero strain is homogeneous. Finally, let f be of the order $(a/r)^3$ to satisfy assumption (i).

In order to incorporate the symmetry properties available by the use of spherical coordinates, imagine that a spherical region of radius R is cut out of the infinite matrix. Also, assume that this region is concentric with the inclusion of radius "a"; see Figure 4. The strain energy stored in this region may be expressed as

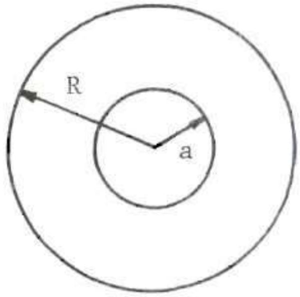

Figure 4.

Concentric Spherical Regions.

$$U(R) = \tfrac{1}{2} \int_D \underline{\underline{S}}_s : \underline{\underline{E}} \, dv \qquad (2\text{-}4)$$

where $\underline{\underline{E}} = \frac{1}{2}(\nabla \underline{u} + \underline{u}\nabla)$. Note that for $\underline{u}$ homogeneous; i.e., no rotation and hence $\underline{\omega} = 0$; $\underline{\underline{E}} = \frac{1}{3} \vartheta \, \underline{\underline{I}} = \nabla \underline{u}$. Furthermore, $(\underline{\underline{I}} : \nabla\underline{u})\underline{\underline{I}} = (\nabla \cdot \underline{u})\underline{\underline{I}} = \vartheta \, \underline{\underline{I}}$. Hence, (2-4) may be written using Gauss' Theorem as

$$U(R) = \tfrac{1}{2} \int_B \underline{\underline{S}}_s : u_r \underline{\underline{I}} \, dg \quad . \qquad (2\text{-}5)$$

Where u_r is the scalar component of $\underline{u}$. Substituting (2-2) and (2-3) into (2-5) gives

$$U(R) = \tfrac{1}{2} \int_D [(\underline{\underline{S}}_{os} + \underline{\underline{S}}_s') : (u_{ro} + u_r') \, \underline{\underline{I}}] dg \qquad (2\text{-}6)$$

$$U(R) = \tfrac{1}{2} \int_B [\underline{\underline{S}}_{os} : u_{ro} \underline{\underline{I}} + \underline{\underline{S}}_{os} : u_r' \underline{\underline{I}} + \underline{\underline{S}}_s' : u_{ro} \underline{\underline{I}} + \underline{\underline{S}}_s' : u_r' \underline{\underline{I}}] dg \ . \qquad (2\text{-}7)$$

Since the first term in (2-7) is due only to the undisturbed field whereas the other terms are due to the perturbation effect of the inclusion, (2-7) may be expressed as

$$U(R) = U_o(R) + \Delta U \tag{2-8}$$

where

$$U_o(r) = \frac{1}{2} \int_B \underline{\underline{S}}_{os} : u_{ro} \underline{\underline{I}} \, dg \tag{2-9}$$

$$\Delta U = \frac{1}{2} \int_B [\underline{\underline{S}}_{os} : u_r{}' \underline{\underline{I}} \quad \underline{\underline{S}}_s{}' : u_{ro} \underline{\underline{I}}] dg \tag{2-10}$$

the term $\underline{\underline{S}}_s{}' : u_r{}' \underline{\underline{I}}$ being of the order $(a/R)^6$ has been omitted.

Designate by $\underline{\underline{E}}^* = \epsilon^*_{ij} \hat{i}_i \hat{i}_j$ the gross strains in the dispersed system. Let there be "c" spheres dispersed in the matrix, the centers of which are located at the points $\underline{X}^{(c)}$. See Figure 5. Now by assumption (iii) the displacement at any point A in the dispersed system may be written as

$$\underline{u} = \underline{u}_o + \sum_{j=1}^{C} \underline{u}_j'^{(c)} \tag{2-11}$$

where the $\underline{u}_j'^{(c)}$ have been defined in (2-2). Therefore

$$\underline{\underline{E}}^* = \nabla \underline{u} = \underline{\underline{E}}_o + \sum_{j=1}^{C} \nabla_x \cdot \underline{u}_j'^{(c)} \underline{\underline{I}} \, . \tag{2-12}$$

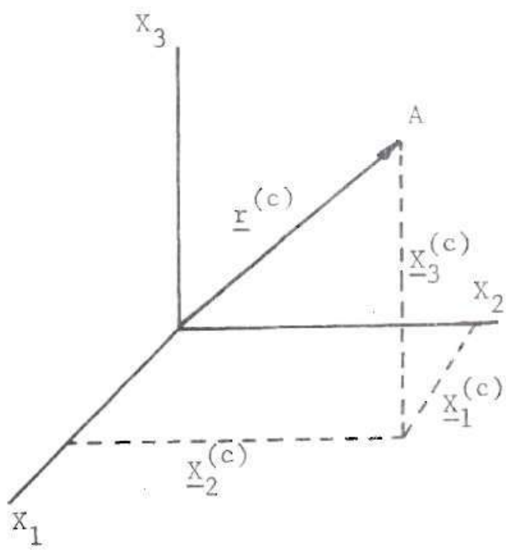

Figure 5.

Position Vector of a Typical Particle.

In order to get the gross strain due to the perturbational effects into a more convenient form, denote by $\underline{r}^{(c)}$ the position vector from the center of a spherical inclusion to point A.

Then, $|\underline{r}^{(c)}|$ is given by Figure 5.

$$\underline{r}^{(c)} \cdot \underline{r}^{(c)} = (\underline{X} - \underline{X}^{(c)}) \cdot (\underline{X} - \underline{X}^{(c)}) \tag{2-13}$$

where

$$\underline{X} = X_{\alpha} \underline{i}_{\alpha}$$

and

$$\underline{X}^{(c)} = X_{\alpha}^{(c)} \underline{i}_{\alpha} \; .$$

Introducing the coordinates

$$\underline{y}^{(c)} = \underline{X} - \underline{X}^{(c)} \tag{2-14}$$

and transforming (2-12) gives

$$\underline{\underline{E}}^{*} = \underline{\underline{E}}_{o} + \sum_{j=1}^{C} \nabla_{y} \cdot \underline{u}_{j}^{\prime (c)} \underline{\underline{I}} \; . \tag{2-15}$$

Choosing, for convenience, A as the origin of coordinates; $\underline{X} = 0$ and $\underline{y}^{(c)} = -\underline{X}^{(c)}$ making (2-15)

$$\underline{\underline{E}}^{*} = \underline{\underline{E}}_{o} - \sum_{j=1}^{C} \nabla_{X^{c}} \cdot \underline{u}_{j}^{\prime (c)} \underline{\underline{I}} \; . \tag{2-16}$$

Designating by "N" the number of particles per unit volume of the system, (2-16) may be expressed as

$$\underline{\underline{E}}^{*} = \underline{\underline{E}}_{o} - N \int_{D} \nabla_{X^{c}} \cdot \underline{u}^{\prime} \underline{\underline{I}} \, dv \tag{2-17}$$

where D is the volume of the entire dispersed system, and the summation has been replaced by volume integration because of the large number of inclusions. By an application of Gauss' Theorem (2-17) may be transformed into a surface integral

$$\int_D \nabla_{x^c} \cdot \underline{u}' \underline{\underline{I}} \, dv = \int_B \underline{u}' \cos(\underline{u}', \underline{r}) \, dg$$

$$= \int_B \underline{u}' \cdot (\underline{X}/r) \, dg \tag{2-18}$$

where B is the surface of the region D. The use of (2-18) transforms (2-17) into

$$\underline{\underline{E}}^* = \underline{\underline{E}}_o - N \int_B \underline{u}' \cdot (\underline{X}/r) \, dg \quad . \tag{2-19}$$

Recalling that

$$f = \frac{(c) \frac{4}{3} \pi a^3}{\frac{4}{3} \pi R^3}$$

and

$$N = \frac{C}{\frac{4}{3} \pi R^3} ,$$

(2-19) may readily be expressed as

$$\underline{\underline{E}}^* = \underline{\underline{E}}_o - \frac{f}{\frac{4}{3} \pi a^3} \int_B \underline{u}' \cdot (\underline{X}/r) \, dg \tag{2-20}$$

where "a" is the radius of a particle.

In a manner analogous to the previous discussion allow the region D to be a sphere of radius R and let there be "c" number of particles within this sphere. Therefore, the strain energy stored in D is

$$U^*(R) = U_o(R) + \sum_{i=1}^{c} \Delta U = U_o(R) + U'(R) \tag{2-21}$$

where $U_o(R)$ and ΔU have been defined by (2-9) and (2-10). $U^*(R)$ may also be computed by considering the dispersed system as a homogeneous, isotropic, and elastic material. The expression for $U^*(R)$ will involve the gross material constants $\varkappa^*$, μ^*, and the strains $\underline{\underline{E}}^*$.

The $\underline{\underline{E}}^*$ may be expressed in terms of known quantities from (2-20). Choosing for the unperturbed field $\underline{u}_o$ values for $\underline{\underline{S}}_{os}$ of isotropic stress and pure shear respectively, the expression for $U^*(R)$ will contain $\varkappa^*$ only in the first case and μ^* only in the second. Thus, one may obtain two equations which determine $\varkappa^*$ and μ^*.

Rewrite the perturbational part of the displacement as

$$\underline{u}' = \rho \underline{u}'' \tag{2-22}$$

where ρ contains the ratio between the elastic moduli of the particle and the medium; while $\underline{u}''$ is independent of this ratio. Once the displacements are known the stresses may readily be computed from Hooke's Law

$$\underline{\underline{S}} = \lambda \vartheta \, \underline{\underline{I}} + 2\mu \, \underline{\underline{E}} \, . \tag{2-23}$$

Obviously, one may conclude that

$$\underline{\underline{S}}' = \rho \underline{\underline{S}}'' \quad . \tag{2-24}$$

Also, analogous to (2-10)

$$\Delta U'' = \frac{1}{2} \int_B [\underline{\underline{S}}_o : u_r'' \underline{\underline{I}} + \underline{\underline{S}}'' : u_{ro} \underline{\underline{I}}] \, dg \quad . \tag{2-25}$$

Thus

$$\Delta U = \rho \Delta U'' \quad \text{and} \quad U'(R) = \rho U''(R) \quad . \tag{2-26}$$

If "ω" is determined from the surface integral involved; i.e.,

$$\omega = \left(\frac{1}{\underline{\underline{E}}_o}\right) \frac{1}{\frac{4}{3} \pi a^3} \int_B \underline{u}' \cdot \left(\frac{\underline{X}_j}{r}\right) dg$$

then, (2-20) may be expressed as

$$\underline{\underline{E}}^* = \underline{\underline{E}}_o \, (1 - \omega f) \quad . \tag{2-27}$$

Since the integral in (2-20) is proportional to $\underline{u}'$, (2-27) may be written as

$$\underline{\underline{E}}^* = \underline{\underline{E}}_o (1 - \rho \omega'' f) \quad . \tag{2-28}$$

Equation (2-21) may be rewritten as

$$\frac{U^*(R)}{U_o(R)} = 1 + \frac{U'(R)}{U_o(R)} \quad . \tag{2-29}$$

Now, choose for the unperturbed field a state of homogeneous stress or strain. Choosing isotropic stress and strain which is defined by

$$\underline{\underline{S}} = \sigma_o \underline{\underline{I}} \; ; \qquad \underline{\underline{E}} = \epsilon_o \underline{\underline{I}} \tag{2-30}$$

where

$$\epsilon_o = \frac{\sigma_o}{3\varkappa}$$

(from Hooke's Law for an isotropic elastic material). Thus, from (2-9) one readily obtains

$$U_o(R) = \tfrac{1}{2} \int_B (3\varkappa_m \underline{\underline{E}}:u_{ro}\underline{\underline{I}}) \; dg \tag{2-31}$$

which is by use of Gauss' theorem

$$U_o(R) = \tfrac{1}{2} \int_D 3\varkappa_m (\underline{\underline{E}}:\underline{\underline{E}}) \; dv \tag{2-32}$$

$$= (3/2)\varkappa_m (\underline{\underline{E}}:\underline{\underline{E}}) D$$

where $\varkappa_m$ is the bulk modulus of the matrix. Similarly

$$\left(\frac{\epsilon^*}{\epsilon_o}\right)^2 \frac{\varkappa^*}{\varkappa_m} = 1 + \rho \, \frac{U''(R)}{U_o(R)} \; . \tag{2-34}$$

Looking at (2-28) and neglecting the f^2 terms

$$\left(\frac{\epsilon^*}{\epsilon_o}\right)^2 = 1 - 2 \; \rho\omega'' f \; . \tag{2-35}$$

Finally, substitution of (2-35) into (2-34) yields

$$\varkappa_{rel} = \frac{\varkappa^*}{\varkappa_m} = 1 + \rho \left[\frac{U''(R)}{U_o(R)} + 2\omega''\rho f\right] \tag{2-36}$$

Hence, (2-36) is a first order approximation of $\varkappa^*$ where the ratio between the moduli of the particle and matrix material enters only through ρ.

The same argument holds for any unperturbed homogeneous state of stress or strain and may be used in the case of pure shear to obtain μ^*. The method described above to determine $\varkappa^*$ and μ^* is that due primarily to Hashin (21). From (2-36) he made the following observation: if a relative gross elastic modulus M_{rel} is known for a specific ratio between a modulus of the particle and the matrix $\tau = M_p/M_m$ and is given by

$$M_{rel} = \frac{M^*}{M_m} = 1 + a_1 f$$

then, for any other ratio τ'

$$M_{rel} = 1 + \left(\frac{\rho}{\rho'}\right) a_1 f \qquad (2\text{-}37)$$

where $\rho = \rho(\tau)$ and $\rho' = \rho'(\tau')$.

As an illustrative example Hashin demonstrated how to actually solve (2-36). The method of attack is to choose isotropic stress as the homogeneous state which may be expressed in terms of a purely radial displacement of the form

$$\underline{u}_o = \epsilon_o \, \underline{r} \quad . \qquad (2\text{-}38)$$

The general solution for radial symmetry may be found in most books on elasticity (14) and is given by

$$\underline{u}_r = Ar\hat{r} + B/r^2 \hat{r} \qquad (2\text{-}39)$$

and

$$\underline{\underline{S}}_r = \sigma_r \, \hat{r}\hat{r} = \left(3\varkappa A - \frac{4\mu B}{r^3}\right) \hat{r}\hat{r} \tag{2-40}$$

where A and B are constants.

The solution to the boundary value problem has two distinct segments: the region inside and outside the inclusion. See Figure 6.

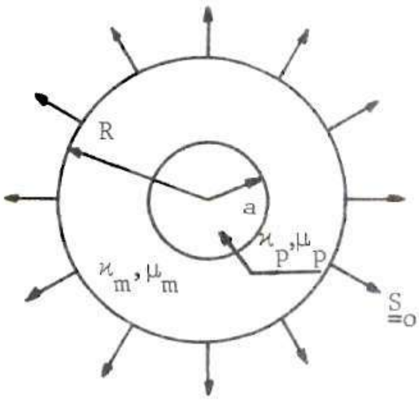

Figure 6.

Radial Tension.

Therefore

$$\underline{u}_{r(p)} = \left(A_{(p)} r + \frac{B_{(p)}}{r^2}\right)\hat{r} \tag{2-41}$$

$$\underline{\underline{S}}_{r(p)} = \left(3\varkappa_p A_{(p)} - \frac{4\mu_p B_{(p)}}{r^3}\right)\hat{r}\hat{r}$$

$$\underline{u}_{r(m)} = \left(A_{(m)} r + \frac{B_{(m)}}{r^2}\right)\hat{r} \tag{2-42}$$

$$\underline{\underline{S}}_{r}(m) = \left(3\varkappa_m A_{(m)} - \frac{4\mu_m B_{(m)}}{r^3}\right)\hat{r}\hat{r} \; .$$

The boundary conditions are

$$\underline{u}_{r(m)} = \epsilon_o \, \underline{r} \; , \quad \underline{r} \to \infty \tag{2-43}$$

$$\underline{u}_{r(p)} = 0 \quad , \quad \underline{r} = 0$$

$$\left.\begin{aligned} \underline{u}_{r(m)} &= \underline{u}_{r(p)} \\ \underline{\underline{S}}_{r(m)} &= \underline{\underline{S}}_{r(p)} \end{aligned}\right\} \; \underline{r} = a\hat{r} \; .$$

These four boundary conditions uniquely determine the four constants in

(2-41) and (2-42). Thus, the solution for the boundary value problem described earlier for the case of isotropic stress is attained. Hashin gives the following results

$$\underline{u}_{r(m)} = \epsilon_o \left(1 + \frac{1 - \frac{\varkappa_p}{\varkappa_m}}{\frac{4\mu_m}{3\varkappa_m} + \frac{\varkappa_p}{\varkappa_m}} \frac{a^3}{r^3} \right) \underline{r} \tag{2-44}$$

since

$$\underline{u}_{ro(m)} = \epsilon_o \underline{r} \ .$$

Hence, from (2-22)

$$\rho = \frac{1 - \frac{\varkappa_p}{\varkappa_m}}{\frac{4\mu_m}{3\varkappa_m} + \frac{\varkappa_p}{\varkappa_m}} \ . \tag{2-45}$$

For a system with rigid particles, Hashin (7) showed that

$$\frac{\varkappa^*}{\varkappa_m} = 1 + 3 \frac{1-\nu_m}{1+\nu_m} f \ . \tag{2-46}$$

Here $\varkappa_p \to \infty$ and $\rho' = -1$. Therefore, using (2-37), (2-45), and (2-46)

$$\varkappa_{rel} = \frac{\varkappa^*}{\varkappa_m} = 1 - 3(1-\nu_m) \frac{1 - \frac{\varkappa_p}{\varkappa_m}}{2(1-2\nu_m) + (1+\nu_m) \frac{\varkappa_p}{\varkappa_m}} f \ . \tag{2-47}$$

A similar development for pure shear yields

$$\frac{\mu^*}{\mu_m} = 1\text{-}15(1-\nu_m)\ \frac{1-\dfrac{\mu_p}{\mu_m}}{7-5\nu_m + 2(4-5\nu_m)\dfrac{\mu_p}{\mu_m}}\ f\ . \qquad (2\text{-}48)$$

Recall the example used in Section I. So doing the following quantities are known

$\mu_m = 3.8 \times 10^6$ psi $\qquad \mu_p = 12 \times 10^6$ psi

$\nu_m = 0.33 \qquad \nu_p = 0.25$

$\kappa_m = 9.8 \times 10^6$ psi $\qquad \kappa_p = 20 \times 10^6$ psi .

Choose $f = 0.05$ which is equivalent to assuming a dilute suspension; i.e., dispersed system. Therefore

$$\kappa^*_{II} = (9.8 \times 10^6)\left[1\text{-}3(0.67)\ \frac{1-\left(\frac{20}{9.8}\right)}{(2)(0.33) + (1.33)\left(\frac{20}{9.8}\right)}(0.05)\right]$$

$$\kappa^*_{II} = 10.1 \times 10^6 \text{ psi}$$

$$\mu^*_{II} = (3.8 \times 10^6)\left[1\text{-}15(0.67)\ \frac{1-\left(\frac{12}{3.8}\right)}{7-(5)(0.33) + 2(4-1.66)\left(\frac{12}{3.8}\right)}\ (0.05)\right]$$

$$\mu^*_{II} = 4.0 \times 10^6 \text{ psi}$$

$$\frac{3}{E^*} = \frac{1}{\mu^*} + \frac{1}{3\kappa^*} = \left(\frac{1}{4.0} + \frac{1}{(3)(10.1)}\right) 10^{-6}$$

$$\therefore \qquad E^*_{II} = 10.6 \times 10^6 \text{ psi}\ .$$

From Table I

$$E^*_L = 10.35 \times 10^6 \text{ psi} ,$$

$$E^*_u = 11.04 \times 10^6 \text{ psi} ,$$

$$E^*_I = \frac{E^*_L + E^*_u}{2} = 10.68 \times 10^6 \text{ psi} .$$

Hence, $E^*_{II} \approx \bar{E}^*_I$ for dispersed systems. However, this result was expected since E^*_{II} is an attempt at an exact result and the method of Chapter I was shown to be reliable for small volume concentrations.

CHAPTER III

AN EXACT GENERAL APPROACH

The previous discussions and analysis have dealt with an aelotropic media composed of a matrix of material one and small volume concentrations of inclusions of material two. Also, the inclusions were restricted to be spherical in shape. In a general form, however, a heterogeneous media may be composed of a matrix and finite volume concentrations of inclusions which are of arbitrary shape and size.

The present analysis will not attempt to describe the internal stress state of the media in detail since this is, as yet, a hopelessly complex task. The dependence of the over-all macroscopic elastic moduli of the composite on the relative concentrations of the constituents, the distribution and size of the inclusions, and the geometrical arrangement of the inclusions in the matrix will be investigated. The restrictions placed on the composite's constituents are that they are homogeneous and isotropic. The heterogeneous media itself is assumed to be macroscopically homogeneous.

Basic Relationships

As the two fundamental independent isotropic elastic constants choose the bulk modulus κ and the shear modulus μ (as was done previously). The dependent material moduli are the stiffness E and Poisson's ratio ν. The following relationships are well known

$$\nu = \frac{\left(\frac{1}{2} - \frac{\mu}{3\kappa}\right)}{\left(1 + \frac{\mu}{3\kappa}\right)} \tag{3.1}$$

$$\frac{3}{E} = \frac{1}{\mu} + \frac{1}{3\kappa}$$

$$E = 2\mu(1+\nu) = 3\kappa(1-2\nu)$$

As done above, the moduli of the two constituent phases will be subscripted by 1 and 2; however, it will be unnecessary to distinguish which subscript refers to which phase except in cases where the expressions are explicitly dependent on the geometry. In keeping with the previous convention an asterisk will be used to denote the macroscopic moduli of the media.

For this development the volume concentrations of both phases will be used and will be denoted by f_1 and f_2, where $f_1 + f_2 = 1$. Reasons of symmetry make it desirable to retain both f_1 and f_2 instead of eliminating one in favor of the other. Once again $\underline{\underline{S}}$ and $\underline{\underline{E}}$ will stand for the stress and strain tensors respectively.

Average values of quantities will have a bar placed above them. The average value will be defined as being the integral of the quantity in question over a specified region divided by the volume of the region. Hence, for a region which contains the constituent phases in the given volume concentrations, one may express $\overline{\underline{\underline{S}}}^*$ and $\overline{\underline{\underline{E}}}^*$ as follows

$$\overline{\underline{\underline{S}}}^* = f_1\overline{\underline{\underline{S}}}_1 + f_2\overline{\underline{\underline{S}}}_2 \tag{3.2}$$

$$\overline{\underline{\underline{E}}}^* = f_1\overline{\underline{\underline{E}}}_1 + f_2\overline{\underline{\underline{E}}}_2$$

In index form the generalized Hooke's Law may be stated as

$$\sigma_{ij} = C_{ijk\ell}\, e_{k\ell} \tag{3.3}$$

where the $C_{ijk\ell}$ are the eighty-one generalized elastic stiffness moduli. Let $\underset{\sim}{L}$ represent the symmetric fourth order tensor which corresponds to the $C_{ijk\ell}$. In a similar manner, let $\underset{\sim}{M}$ represent the symmetric fourth order tensor of the compliances. The symmetry properties invoked are the following, using $C_{ijk\ell}$ as the scalar component of $\underset{\sim}{L}$

$$C_{ijk\ell} = C_{jik\ell} = C_{ij\ell k} = C_{k\ell ij} \tag{3.4}$$

which reduces the eighty-one independent constants to twenty-one independent constants. Thus, one may note the following relationships

$$\underset{=}{S}_1 = \underset{\sim}{L}_1 : \underset{=}{E}_1 \qquad \underset{=}{S}_2 = \underset{\sim}{L}_2 : \underset{=}{E}_2 \tag{3.5}$$

$$\underset{=}{E}_1 = \underset{\sim}{M}_1 : \underset{=}{S}_1 \qquad \underset{=}{E}_2 = \underset{\sim}{M}_2 : \underset{=}{S}_2 \ .$$

Since the phases have been assumed to be uniform and isotropic, identical relations involving the same quantities also hold between the corresponding averaged quantities. Hence, substituting (3.5) into (3.2) yields

$$\overline{\underset{=}{S}}{}^* = f_1 \underset{\sim}{L}_1 : \overline{\underset{=}{E}}_1 + f_2 \underset{\sim}{L}_2 : \overline{\underset{=}{E}}_2 \tag{3.6}$$

$$\overline{\underset{=}{E}}{}^* = f_1 \underset{\sim}{M}_1 : \overline{\underset{=}{S}}_1 + f_2 \underset{\sim}{M}_2 : \overline{\underset{=}{S}}_2 \ .$$

<u>General Results</u>

If a large sample volume containing a large number of inclusions is considered, then the macroscopic moduli of the composite may be assumed to be independent of the surface tractions and displacements. Thus there should be a unique dependence of the average strains in the constituents upon the over-all macroscopic strains. Define the relation as

$$\bar{\underline{\underline{E}}}_1 = \underset{\sim}{A}_1 : \bar{\underline{\underline{E}}}^* \tag{3.7}$$

$$\bar{\underline{\underline{E}}}_2 = \underset{\sim}{A}_2 : \bar{\underline{\underline{E}}}^*$$

where $f_1\underset{\sim}{A}_1 + f_2\underset{\sim}{A}_2 = \underset{\sim}{J}$. In (3.7) $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are tensors dependent on the volume concentrations and phase moduli and are generally unsymmetric. The tensor $\underset{\sim}{J}$ is the basic isotropic tensor of the fourth order and exhibits the following property

$$\underset{\sim}{J} : \underline{\underline{R}} = \underline{\underline{R}} \ . \tag{3.8}$$

Now, if (3.7) is combined with the first of (3.6), one has

$$\bar{\underline{\underline{S}}}^* = \underset{\sim}{L}^* : \bar{\underline{\underline{E}}}^* \tag{3.9}$$

where $\underset{\sim}{L}^* = f_1\underset{\sim}{L}_1 : \underset{\sim}{A}_1 + f_2\underset{\sim}{L}_2 : \underset{\sim}{A}_2$ and is the fourth order stiffness tensor of the composite moduli. Similarly, one may write for the stresses

$$\bar{\underline{\underline{S}}}_1 = \underset{\sim}{B}_1 : \bar{\underline{\underline{S}}}^* \tag{3.10}$$

$$\bar{\underline{\underline{S}}}_2 = \underset{\sim}{B}_2 : \bar{\underline{\underline{S}}}^*$$

where $f_1\underset{\sim}{B}_1 + f_2\underset{\sim}{B}_2 = \underset{\sim}{J}$. Hence, substituting (3.10) into the second of (3.7) yields

$$\bar{\bar{E}}^* = \underset{\sim}{M}^* : \bar{\bar{S}}^* \tag{3.11}$$

where $\underset{\sim}{M}^* = f_1\underset{\sim}{M}_1:\underset{\sim}{B}_1 + f_2\underset{\sim}{M}_2:\underset{\sim}{B}_2$ and is the fourth order tensor of the compliances of the composite material.

The beauty of this approach may be shown by the following manipulations. Suppose that the average value of the stress or strain in phase one can be determined for the arbitrary over-all macroscopic values; then, $\underset{\sim}{B}_1$ or $\underset{\sim}{A}_1$ respectively would be determined from (3.10) or (3.7). Now, set (3.8) equal to the first of (3.6)

$$\bar{\bar{S}}^* = \underset{\sim}{L}^*:\bar{\bar{E}}^* = f_1\underset{\sim}{L}_1:\bar{\bar{E}}_1 + f_2\underset{\sim}{L}_2:\bar{\bar{E}}_2 \tag{3.12}$$

substituting (3.7) into (3.12)

$$\underset{\sim}{L}^*:\bar{\bar{E}}^* = f_1\underset{\sim}{L}_1:(\underset{\sim}{A}_1:\bar{\bar{E}}^*) + f_2\underset{\sim}{L}_2:(\underset{\sim}{A}_2:\bar{\bar{E}}^*) \tag{3.13}$$

$$= (f_1\underset{\sim}{L}_1:\underset{\sim}{A}_1 + f_2\underset{\sim}{L}_2:\underset{\sim}{A}_2):\bar{E}^* \quad .$$

Hence

$$\underset{\sim}{L}^* = f_1\underset{\sim}{L}_1:\underset{\sim}{A}_1 + f_2\underset{\sim}{L}_2:\underset{\sim}{A}_2 \quad . \tag{3.14}$$

Since A_1 has been assumed known, one has

$$f_2\underset{\sim}{A}_2 = \underset{\sim}{J} - f_1\underset{\sim}{A}_1 \ . \tag{3.15}$$

Substituting (3.15) into (3.14) gives

$$\underset{\sim}{L}^* = f_1\underset{\sim}{L}_1:\underset{\sim}{A}_1 + \underset{\sim}{L}_2:(\underset{\sim}{J} - f_1\underset{\sim}{A}_1) \tag{3.16}$$

$$= f_1(\underset{\sim}{L}_1 - \underset{\sim}{L}_2):\underset{\sim}{A}_1 + \underset{\sim}{L}_2 \ .$$

Therefore

$$\underset{\sim}{L}^* - \underset{\sim}{L}_2 = f_1(\underset{\sim}{L}_1 - \underset{\sim}{L}_2):\underset{\sim}{A}_1 \qquad (3.17)$$

(3.17) or (3.16) determine $\underset{\sim}{L}^*$ completely since f_1, $\underset{\sim}{L}_1$, $\underset{\sim}{L}_2$, and $\underset{\sim}{A}_1$ are all known quantities. Interchanging phase one with phase two in (3.17) gives $\underset{\sim}{L}^*$ for $\underset{\sim}{A}_2$ known. An analogous development using (3.11), the second of (3.6), (3.10), and $f_2\underset{\sim}{B}_2 = \underset{\sim}{J} - f_1\underset{\sim}{B}_1$ yields

$$\underset{\sim}{M}^* - \underset{\sim}{M}_2 = f_1(\underset{\sim}{M}_1 - \underset{\sim}{M}_2):\underset{\sim}{B}_1 \quad . \qquad (3.18)$$

Thus, the composite compliances M* may be explicitly determined if either $\underset{\sim}{B}_1$ or $\underset{\sim}{B}_2$ can be determined.

On the other hand the converse relationships may be found. Assume that $\underset{\sim}{L}^*$ and $\underset{\sim}{M}^*$ are known - perhaps from some design criterion. Then, from (3.5) and (3.10) one may obtain

$$\underset{=}{\bar{S}}_1 = \underset{\sim}{L}_1:\underset{=}{\bar{E}}_1 = \underset{\sim}{B}_1:\underset{=}{\bar{S}}^* \ . \qquad (3.19)$$

Substituting (3.7) and (3.9) into (3.19) yields

$$\underset{\sim}{L}_1:(\underset{\sim}{A}_1:\underset{=}{\bar{E}}^*) = \underset{\sim}{B}_1:(\underset{\sim}{L}^*:\underset{=}{\bar{E}}^*)$$

$$(\underset{\sim}{L}_1:\underset{\sim}{A}_1):\underset{=}{\bar{E}}^* = (\underset{\sim}{B}_1:\underset{\sim}{L}^*):\underset{=}{\bar{E}}^*$$

$$\underset{\sim}{L}_1:\underset{\sim}{A}_1 = \underset{\sim}{B}_1:\underset{\sim}{L}^* \ . \qquad (3.20)$$

Similarily, from (3.5), (3.10), (3.7), and (3.11) one gets

$$\underset{\sim}{A}_1:\underset{\sim}{M}^* = \underset{\sim}{M}_1:\underset{\sim}{B}_1 \ . \qquad (3.21)$$

Relations similar to (3.20) and (3.21) may be obtained for the other subscript. Hence, if $\underset{\sim}{L}^*$, $\underset{\sim}{M}^*$, and say $\underset{\sim}{A}_1$ are known the moduli of the different constituents may be determined from (3.20), (3.21) and the analogous relations for the other subscript.

The merit of this approach to the general composite material problem is its mathematical beauty and simplicity. Nevertheless, in actual practice, the method may prove to be somewhat difficult to apply. The exact dependence of $\underset{\sim}{A}_1$, $\underset{\sim}{A}_2$, $\underset{\sim}{B}_1$, and $\underset{\sim}{B}_2$ on the volume concentrations and phase moduli has not been stated. These fourth order tensors can be determined if the values of the average strain fields in the separate phases and the over-all composite are known by using equations (3.7) and (3.10). Still, in their most general form, $\underset{\sim}{L}^*$ and $\underset{\sim}{M}^*$ contain twenty-one independent constants each, and the labor involved in determining these constants may prove to be almost insurmountable. If the heterogeneous media is itself assumed to be isotropic in the large, then the above method reduces to a very useful tool.

Isotropic Reduction

If not only are the constituent phases isotropic, but the macroscopic effects of the heterogeneous material are such that it, too, may be considered isotropic; then, the previous analysis will be greatly simplified. The number of independent constants to be determined will be reduced from twenty-one to two; more directly the bulk modulus κ and the shear modulus μ. Thus, in order to determine these two constants only two independent strain fields need to be considered.

Case of Pure Shear

Assume that the composite specimen is subjected to a pure shear strain, say $\bar{\underline{\underline{E}}}*_{\gamma}$, its plane and direction being arbitrary. The over-all stress is also a pure shear, say $\bar{\underline{\underline{S}}}*_{\tau}$. Since the mixture is assumed isotropic the following relation is apparent

$$\bar{\underline{\underline{S}}}*_{\tau} = 2\mu*\bar{\underline{\underline{E}}}*_{\gamma} . \tag{3.22}$$

This also follows from (3.12). For this particular case the following are true.

$$\underset{\sim}{L}_1 = 2\mu_1 \qquad \underset{\sim}{M}_1 = \tfrac{1}{2}\left(\frac{1}{\mu_1}\right) \tag{3.23}$$

$$\underset{\sim}{L}_2 = 2\mu_2 \qquad \underset{\sim}{M}_2 = \tfrac{1}{2}\left(\frac{1}{\mu_2}\right)$$

$$\underset{\sim}{L}* = 2\mu* \qquad \underset{\sim}{M}* = \tfrac{1}{2}\left(\frac{1}{\mu*}\right) .$$

Equation (3.6) may now be written as

$$\bar{\underline{\underline{S}}}*_{\tau} = f_1\mu_1\bar{\underline{\underline{E}}}_{1\gamma} + f_2\mu_2\bar{\underline{\underline{E}}}_{2\gamma} \tag{3.24}$$

$$\bar{\underline{\underline{E}}}*_{\gamma} = \frac{f_1\bar{\underline{\underline{S}}}_{1\tau}}{\mu_1} + \frac{f_2\bar{\underline{\underline{S}}}_{2\tau}}{\mu_2} .$$

Here $\bar{\underline{\underline{S}}}_{1\tau}$ and $\bar{\underline{\underline{S}}}_{2\tau}$ are the deviatoric parts of the stress while $\bar{\underline{\underline{E}}}_{1\gamma}$ and $\bar{\underline{\underline{E}}}_{2\gamma}$ are the deviatoric parts of the strain. All of these are averaged over the respective phases. Equation (3.7) may be rewritten as

$$\bar{\underline{\underline{E}}}_{1\gamma}:(\bar{\underline{\underline{E}}}*_{\gamma})^{-1} = a_1 \tag{3.25}$$

$$\bar{\underline{\underline{E}}}_{2\gamma}:(\bar{\underline{\underline{E}}}*_{\gamma})^{-1} = a_2$$

with $a_1f_1 + a_2f_2 = 1$. Similarly, (3.10) becomes

$$\underline{\underline{S}}_{1_T} : (\overline{\underline{\underline{S}}}^*{}_T)^{-1} = b_1 \tag{3.26}$$

$$\underline{\underline{S}}_{2_T} : (\overline{\underline{\underline{S}}}^*{}_T)^{-1} = b_2$$

with $b_1f_1 + b_2f_2 = 1$. The a's and the b's are functions of the concentrations and phase moduli. (3.14) now becomes

$$\mu^* = f_1\mu_1a_1 + f_2\mu_2a_2 \tag{3.27}$$

and $\underset{\sim}{M}^* = f_1\underset{\sim}{M}_1 : \underset{\sim}{B}_1 + f_2\underset{\sim}{M}_2 : \underset{\sim}{B}_2$ is reduced to

$$\frac{1}{\mu^*} = \frac{f_1b_1}{\mu_1} + \frac{f_2b_2}{\mu_2} \ . \tag{3.28}$$

Finally, (3.17) and (3.18) may be expressed as

$$\frac{\mu^* - \mu_2}{\mu_1 - \mu_2} = f_1a_1 \qquad \frac{\left(\frac{1}{\mu^*}\right) - \left(\frac{1}{\mu_2}\right)}{\left(\frac{1}{\mu_1}\right) - \left(\frac{1}{\mu_2}\right)} = b_1f_1 \ . \tag{3.29}$$

Equations (3.27), (3.28), and (3.29) are all equivalent expressions for the overall shear modulus. The equivalence may be seen by interchanging a_1 and b_1 by using $\mu^* = \frac{a_1\mu_1}{b_1}$ which is the scalar form of (3.20). Write (3.27) as

$$\frac{1}{\mu^*} = \frac{f_1\mu_1a_1}{\mu^{*2}} + \frac{f_2\mu_2a_2}{\mu^{*2}} \tag{3.30}$$

$$\frac{1}{\mu^*} = \frac{f_1b_1^2}{a_1\mu_1} + \frac{f_2b_2^2}{a_2\mu_2}$$

$$\frac{1}{\mu^*} = \frac{f_1 b_1}{\mu_1}\left(\frac{b_1}{a_1}\right) + \frac{f_2 b_2}{\mu_2}\left(\frac{b_2}{a_2}\right) .$$

However, from $f_1 \underset{\sim}{A}_1 + f_2 \underset{\sim}{A}_2 = \underset{\sim}{\vartheta} = f_1 \underset{\sim}{B}_1 + f_2 \underset{\sim}{B}_2$ the following is shown.

$$f_1 a_1 + f_2 a_2 = f_1 b_1 + f_2 b_2$$

$$f_1(a_1 - b_1) + f_2(a_2 - b_2) = 0 \quad .$$

Now, since f_1 and f_2 are assumed positive, one has

$$a_1 = b_1 \quad \Longleftrightarrow \quad \frac{b_1}{a_1} = 1$$

$$a_2 = b_2 \quad \Longleftrightarrow \quad \frac{b_2}{a_2} = 1 .$$

Hence, (3.30) does show the equivalence of (3.27) and (3.28).

Expressions for the overall bulk modulus similar to (3.27), (3.28), and (3.29) may be derived by assuming a strain field on the composite specimen that is a pure dilation, say $\bar{\vartheta}^*$. Hill (10) gives these relations as

$$\varkappa^* = a_1 f_1 \varkappa_1 + a_2 f_2 \varkappa_2 \tag{3.31}$$

$$\frac{1}{\varkappa^*} = \frac{b_1 f_1}{\varkappa_1} + \frac{b_2 f_2}{\varkappa_2}$$

$$\frac{\varkappa^* - \varkappa_2}{\varkappa_1 - \varkappa_2} = a_1 f_1$$

$$\frac{\left(\frac{1}{\varkappa^*}\right) - \left(\frac{1}{\varkappa_2}\right)}{\left(\frac{1}{\varkappa_1}\right) - \left(\frac{1}{\varkappa_2}\right)} = b_1 f_1$$

CHAPTER IV

THE INCLUSION PROBLEM

All of the previous chapters have dealt with methods of determining the over-all bulk properties of an aelotropic media. Most of the investigations that have been undertaken in the area of aelotropic media have been to develop methods of accurately predicting bulk properties. Nevertheless, predicting the internal stress, strain, and displacement fields of such a media is of great interest and importance. For if these fields can be determined, then the material properties of the media can be exactly determined by relations available from the theory of elasticity.

The work involved in expressing the internal stress, strain, and displacement fields is extremely tedious and involved. Generally the expressions contain integrals that can not be evaluated. Several attempts to solve this problem have been carried out. The most notable and important work having been done by J. D. Eshelby (12, 13). This investigation was concerned with what Eshelby called the transformation problem and the ellipsoidal inclusion problem. He showed how the method of the transformation problem could be applied to the ellipsoid and other second-degree surfaces. However, inclusions in the shape of hyperboloids or paraboloids are not of much practical interest.

The Transformation Problem

Eshelby's "transformation Problem" is that of determining the resultant elastic field inside and outside of a surface B which bounds a region (the inclusion) which is in an infinite homogeneous isotropic elastic medium (the matrix) when the enclosed region undergoes a change of shape and size. This change is one that would be an arbitrary homogeneous strain $\underline{\underline{E}}^T$ were it not for the constraint imposed on the changing region by virtue of its confinement in the matrix. As was done for all previous discussions the inclusion is assumed to be perfectly bonded to the matrix material so that no relative displacements occur.

The motivating idea behind the solution of this problem is as follows. Remove the inclusion from the matrix by cutting the media around the surface B. Allow the inclusion to undergo the unconstrained transformation while it is free from the matrix. Apply surface tractions to the transformed inclusion chosen so as to restore it to its original form. Now replace the inclusion back in the matrix and rebond along the cut. One now has a media which contains a matrix with zero stress and an inclusion with a known value of internal stress. The surface tractions have become a built-in layer of body force distributed over the matrix and inclusion interface. This unwanted layer of body forces is removed by applying an equal and opposite layer of body forces over the surface B. The displacement induced by this last layer of body forces is the one that is of interest.

Analysis

The following relationships from the theory of linear elasticity for an isotropic medium will be used

$$\underline{\underline{E}} = \tfrac{1}{2}(\nabla\underline{u} + \underline{u}\nabla) \tag{4-1}$$

$$\underline{\underline{S}} = \lambda\vartheta\underline{\underline{I}} + 2\mu\underline{\underline{E}} \ . \tag{4-2}$$

Equation (4-1) is the relation between the strain $\underline{\underline{E}}$ and the elastic displacement $\underline{u}$, and (4-2) is Hooke's Law. It may be convenient to split a second order tensor $\underline{\underline{A}}$ into its scalar and deviatoric parts

$$\underline{\underline{A}} = 1/3\ a\ \underline{\underline{I}} + \underline{\underline{A}}' \tag{4-3}$$

where

$$a = \underline{\underline{A}}:\underline{\underline{I}} \tag{4-4}$$

$$\underline{\underline{A}}' = \underline{\underline{A}} - 1/3\ a\ \underline{\underline{I}}\ .$$

The displacement $\underline{u}$ will be defined using Lagrangian coordinates; i.e., the underformed coordinates. Hence, if one chooses a point $\underline{r}$ in the material with coordinates $\underline{x}(\underline{r})$ and allows the transformation to take place gradually, then, after the transformation is over the point $\underline{r}$ will have moved and will be located by the coordinates $\underline{x}(\underline{r}) + \underline{u}^c(\underline{r})$. Thus, $\underline{u}^c$ is the displacement, and the state of zero displacement is the state of the material before the transformation has occurred. Throughout the analysis quantities with an affix (e.g. $\underline{u}^c$, $\underline{\underline{E}}^c$, $\underline{\underline{S}}^c$) are to be understood to be related by equations (4-1) and (4-2). Therefore, the inclusion and matrix are assumed to behave isotropically.

First, remove the inclusion from the matrix and allow the transformation $\underline{\underline{E}}^T$ to occur. The inclusion and matrix are both in a state of zero stress. In order to restore the inclusion to its original form apply surface tractions $-S^T\cdot\hat{n}$ to its surface, where

$$\underline{\underline{S}}^T = \lambda\vartheta^T\ \underline{\underline{I}} + 2\mu\underline{\underline{E}}^T\ . \tag{4-5}$$

Place the inclusion back in the cavity in the matrix and rebond the interface B. The surface tractions have now become an embedded layer of body forces on each element dB of B by an amount

$$d\underline{P} = - \underline{\underline{S}}^T \cdot \hat{n}\, dB \tag{4-6}$$

where $\hat{n}$ is the unit outward normal to B.

The matrix is still unstressed, but there is a uniform stress $-S^T$ in the inclusion. Also, the media is still in its original position; i.e., $\underline{u}^c$ is zero. In order to remove the unwanted body forces $-d\underline{P}$, apply an equal and opposite layer of body forces $+d\underline{P}$ over B. This operation leaves the body free of external force, but in a state of internal self-stress due to the transformation of the particle; also, the displacement which occurs because of $+d\underline{P}$ is the displacement $\underline{u}^c$.

The displacement at a point M due to a point force $\underline{P}$ at Q was originally solved by Lord Kelvin. This displacement can be expressed as

$$\underline{u}(M) = \underline{\underline{U}} \cdot \underline{P} \tag{4-7}$$

where Love (20)

$$\underline{\underline{U}} = \frac{1}{4\pi\mu} \frac{\underline{\underline{I}}}{r(M,Q)} - \frac{1}{16\pi\mu(1-\nu)} \nabla_M \nabla_M \,[r(M,Q)] \tag{4-8}$$

where the following correspondences with Love are used

$$\frac{\lambda+\mu}{\lambda+2\mu} = \frac{1}{2(1-\nu)}$$

$$r = r(M,Q) = |\underline{r}(Q) - \underline{r}(M)| \; .$$

Eshelby used $|\underline{r}-\underline{r}'|$ to denote the distance between point Q and point M. Using the following identity

$$\nabla_M r^m = mr^{m-1}\,\hat{r} = m\,r^{m-2}\,\underline{r}$$

where $\underline{r} = r\,\hat{r}$ since $\hat{r}$ is an unit vector

$$\nabla_M^2 r^m = \nabla_M \cdot \nabla_M r^m = m\,\nabla_M \cdot (r^{m-2}\,\underline{r})$$

$$= m(m+1)r^{m-2}$$

one sees that for $m = 1$

$$\frac{1}{r(M,Q)} = 1/2\;\nabla_M^2\;r(M,Q)\;. \tag{4-9}$$

Thus, (4-8) may be rewritten as

$$\underline{\underline{U}} = \frac{1}{4\pi\mu}\left(\frac{\underline{\underline{I}}}{2}\,\nabla_M^2 - \frac{1}{4(1-\nu)}\,\nabla_M\nabla_M\right)r(M,Q)\;. \tag{4-10}$$

The displacement $\underline{u}^c$ can now be obtained by replacing $\underline{P}$ in (4-7) with $d\underline{P} = \underline{\underline{S}}^T \cdot \hat{n}\,dB$ and integrating over B

$$\underline{u}^c = \int_B \underline{\underline{U}} \cdot (\underline{\underline{S}}^T\cdot\hat{n})dB \tag{4-11}$$

$$= \int_B \hat{n} \cdot (\underline{\underline{U}}\cdot\underline{\underline{S}}^T)dB\;.$$

By use of Gauss' Theorem

$$\underline{u}^c = \int_D \nabla_Q \cdot (\underline{\underline{U}}\cdot\underline{\underline{S}}^T)dv \tag{4-12}$$

$$= -\int_D \nabla_M \cdot (\underline{\underline{U}}\cdot\underline{\underline{S}}^T)dv$$

since $\nabla_Q = -\nabla_M$. D is the region bounded by B and dv is a differential volume element of D. Using equation (4-10) to express (4-12) yields

$$\underline{u}^c = \int_D \nabla_M \cdot \left\{\frac{1}{4\pi\mu}\left[\frac{1}{4(1-\nu)}\nabla_M\nabla_M - \frac{\underline{\underline{I}}}{2}\nabla_M^2\right] r(M,Q) \cdot \underline{\underline{S}}^T\right\} dv \tag{4-13}$$

$$= \frac{1}{4\pi\mu}\int_D \left\{\frac{1}{4(1-\nu)}\left[\nabla_M \cdot (\nabla_M\nabla_M r(M,Q) \cdot \underline{\underline{S}}^T)\right]\right.$$

$$\left. - \frac{1}{2}\nabla_M \cdot \left[\nabla_M^2\, r(M,Q)\underline{\underline{I}} \cdot \underline{\underline{S}}^T\right]\right\} dv \ .$$

Look at the last part of (4-13)

$$\int_D \frac{1}{2}\nabla_M \cdot \left[\nabla_M^2\, r(M,Q)\underline{\underline{I}} \cdot \underline{\underline{S}}^T\right] dv \tag{4-14}$$

$$= \int_D \frac{1}{2}\left\{\left[\nabla_M \cdot (\nabla_M^2 r(M,Q)\underline{\underline{I}})\right] \cdot \underline{\underline{S}}^T + \nabla_M^2 r(M,Q)\underline{\underline{I}} : \left[\nabla_M \underline{\underline{S}}^T\right]\right\} dv$$

$$= \int_D \frac{1}{2}\left[\nabla_M\nabla_M^2\, r(M,Q)\right] \cdot \underline{\underline{S}}^T\, dv$$

since $\nabla_M \underline{\underline{S}}^T = 0$ because $\underline{\underline{S}}^T$ is a constant. Now, look at the first part of (4-13)

$$\int_D \nabla_M \cdot \left[\nabla_M\nabla_M\ r(M,Q) \cdot \underline{\underline{S}}^T\right] dv \tag{4-15}$$

$$= \int_D \left\{\left[\nabla_M \cdot \nabla_M\nabla_M\ r(M,Q)\right] \cdot \underline{\underline{S}}^T + \left[\nabla_M\nabla_M\ r(M,Q)\right] : \left[\nabla_M \underline{\underline{S}}^T\right]\right\} dv$$

$$= \int_D \nabla_M\nabla_M^2\ r(M,Q) \cdot \underline{\underline{S}}^T\ dv \ .$$

Thus, (4-13) may be rewritten as

$$\underline{u}^c = \int_D \left\{ \frac{1}{16\pi\mu(1-\nu)} \nabla_M \nabla_M^2 \, r(M,Q) \cdot \underline{\underline{S}}^T - \frac{1}{4\pi\mu} \nabla_M \left(\frac{1}{r(M,Q)} \right) \cdot \underline{\underline{S}}^T \right\} dv \quad (4\text{-}16)$$

Let

$$\varphi = \int_D \frac{1}{r(M,Q)} \, dv$$

and

$$\psi = \int_D r(M,Q) dv \; ;$$

thus, (4-16) becomes

$$\underline{u}^c = \frac{1}{16\pi\mu(1-\nu)} (\nabla_M \nabla_M \nabla_M \; \psi) \cdot \underline{\underline{S}}^T - \frac{1}{4\pi\mu} (\nabla_M \varphi) \cdot \underline{\underline{S}}^T \qquad (4\text{-}17)$$

where

$$(\nabla_M \nabla_M \; \psi)_s = \nabla_M^2 \; \psi \; .$$

Note that φ is the ordinary Newtonian (harmonic) potential of attracting matter of unit density filling the volume D bounded by B, and that ψ is the corresponding biharmonic potential. It is evident from (4-9) that

$$\nabla^2 \psi = 2\varphi \qquad (4\text{-}18)$$

$$\nabla^4 \psi = 2\nabla^2 \varphi = \begin{cases} -8\pi \text{ inside B} \\ 0 \text{ outside B} \end{cases} \; .$$

The second relation in (4-18) is true since

$$\nabla^2\varphi = \begin{cases} -4\pi \text{ inside B} \\ 0 \text{ outside B} \end{cases}$$

from the known properties of the Newtonian potential. The displacement $\underline{u}^c$ may be expressed in terms of strains by use of Hooke's Law

$$\underline{u}^c = \frac{1}{16\pi\mu(1-\nu)} (\nabla_m \nabla_m \nabla_m \psi) \cdot (\lambda\vartheta^T \underline{\underline{I}} + 2\mu \underline{\underline{E}}^T) \tag{4-19}$$

$$- \frac{1}{4\pi\mu} \nabla_m \varphi \cdot (\lambda\vartheta^T \underline{\underline{I}} + 2\mu \underline{\underline{E}}^T)$$

$$= \frac{1}{8\pi(1-\nu)} (\nabla_m \nabla_m \nabla_m \psi) \cdot \underline{\underline{E}}^T - \frac{1}{2\pi} \nabla_m \varphi \cdot \underline{\underline{E}}^T - \frac{\nu}{4\pi(1-\nu)} (\nabla_m \varphi) \cdot \vartheta^T \underline{\underline{I}} .$$

In (4-19),

$$(\nabla_m \nabla_m \psi)_s = \nabla_m^2 \psi = 2\varphi ,$$

and

$$\left[1 - \frac{1}{2(1-\nu)}\right] \frac{\lambda}{\mu} = \frac{\nu}{(1-\nu)} .$$

The displacement field $\underline{u}^c$ is the actual displacement field in the matrix and inclusion after the inclusion has been rebonded in the matrix and the embedded layer of body forces have been relaxed. The strain in the matrix and the inclusion is

$$\underline{\underline{E}}^c = \tfrac{1}{2}(\nabla \underline{u}^c + \underline{u}^c \nabla) \quad . \tag{4-20}$$

The matrix has a stress field $\underline{\underline{S}}^c$ in it due to $\underline{\underline{E}}^c$ which may be expressed as

$$\underline{\underline{S}}^c = \lambda\vartheta^c \underline{\underline{I}} + 2\mu \underline{\underline{E}}^c . \tag{4-21}$$

The inclusion also has a stress field $\underline{\underline{S}}^c$, but the inclusion also has a stress field $-\underline{\underline{S}}^T$ induced when the surface tractions were applied while the inclusion was outside of the matrix in order to return it to the form it had before the transformation $\underline{\underline{E}}^T$ occurred. Hence, the stress field $\underline{\underline{S}}^I$ in the inclusion is

$$\underline{\underline{S}}^I = \underline{\underline{S}}^c - \underline{\underline{S}}^T \tag{4-22}$$

$$= \lambda(\vartheta^c - \vartheta^T)\underline{\underline{I}} + 2\mu(\underline{\underline{E}}^c - \underline{\underline{E}}^T) \ .$$

Therefore, knowing the displacement $\underline{u}^c$ and $\underline{\underline{E}}^T$ one may find all of the corresponding internal stress and strain fields of the body.

The Ellipsoidal Inclusion

If the inclusion is bounded by an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

the Newtonian potential φ may be found explicitly. For an interior point MacMillan (22) gives φ as

$$\varphi_i = \pi abc \int_0^\infty \left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{c^2+s}\right) \frac{dS}{[(a^2+s)(b^2+s)(c^2+s)]^{\frac{1}{2}}} \tag{4-23}$$

This integral can be evaluated in terms of four elliptic integrals

$$A_1 = \int_0^\infty \frac{ds}{\alpha(s)} = \frac{2}{(a^2-c^2)^{\frac{1}{2}}} F(\theta, K) \tag{4-24}$$

$$A_2 = \int_0^\infty \frac{ds}{(a^2+s)\alpha(s)} = \frac{2}{(a^2-c^2)^{3/2}} \left(\frac{1}{K^2}\right)[F(\theta,K) - E(\theta,K)] \tag{4-25}$$

$$A_3 = \int_0^\infty \frac{ds}{(b^2+s)\alpha(s)}$$

$$= \frac{2}{(a^2-c^2)^{3/2}}\left[\frac{E(\theta,K)}{K^2(1-K^2)} - \frac{snv\ cnv}{(1-K^2)dnv} - \frac{F(\theta,K)}{K^2}\right] \qquad (4\text{-}26)$$

$$A_4 = \int_0^\infty \frac{ds}{(c^2+s)\alpha(s)}$$

$$= \frac{2}{(a^2-c^2)^{3/2}}\left[\frac{1}{1-K^2}\left(\frac{snv\ dnv}{cnv} - E(\theta,K)\right)\right] \qquad (4\text{-}27)$$

where

$$\alpha(s) = [(a^2+s)(b^2+s)(c^2+s)]^{\frac{1}{2}} \qquad (4\text{-}28)$$

$$K^2 = \frac{a^2-b^2}{a^2-c^2}$$

$$snv = \sqrt{\frac{a^2-c^2}{a^2}}$$

$$cnv = \frac{c}{a}$$

$$dnv = \frac{b}{a} .*$$

Substituting (4-24), (4-25), (4-26), and (4-28) into (4-23) yields, using (4-28) to rearrange terms

$$\varphi_i = \frac{2\pi abc}{(a^2-c^2)^{\frac{1}{2}}}\left\{\left[1 - \frac{x^2}{a^2-b^2} + \frac{y^2}{a^2-b^2}\quad F(\theta,K)\right]\right. \qquad (4\text{-}29)$$

*The author believes that there is a misprint in The Theory of the Potential by W. D. MacMillan, (Dover Publications) on p. 60.

$$+ \left[\frac{x^2}{a^2-b^2} - \frac{(a^2-c^2)\,y^2}{(a^2-b^2)(b^2-c^2)} + \frac{z^2}{(b^2-c^2)}\right] E(\theta,K)$$

$$+ \left[\frac{c^2y^2}{b^2-c^2} - \frac{b^2z^2}{b^2-c^2}\right] \left(\frac{\sqrt{a^2-c^2}}{abc}\right)\Bigg\} .$$

With some more rearranging (4-29) can be reduced to the form Eshelby uses

$$\varphi_i = \tfrac{1}{2}(a^2-x^2)I_a + \tfrac{1}{2}(b^2-y^2)I_b + \tfrac{1}{2}(c^2-z^2)I_c \tag{4-30}$$

where

$$I_a = \frac{4\pi abc}{(a^2-b^2)(a^2-c^2)^{\frac{1}{2}}} \left[F(\theta,K) - E(\theta,K)\right] \tag{4-31}$$

$$I_b = 4\pi - I_a - I_c$$

$$I_c = \frac{4\pi abc}{(b^2-c^2)(a^2-c^2)^{\frac{1}{2}}} \left[\frac{b(a^2-c^2)^{\frac{1}{2}}}{ac} - E(\theta,K)\right] .^*$$

For an exterior point the form of φ is the same as (4-23) except that the lower limit of integration is not zero.

$$\varphi_e = \pi abc \int_{\varkappa}^{\infty} \left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{c^2+s}\right) \frac{ds}{\alpha(s)} \tag{4-32}$$

where

$$a^2 > b^2 > c^2$$

$\varkappa$ = largest positive root of

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*The author believes that there is a misprint in Progress in Solid Mechanics: Volume II, p. 105. (See Literature Cited - No. 13).

$$\frac{x^2}{a^2+\varkappa} + \frac{y^2}{b^2+\varkappa} + \frac{z^2}{c^2+\varkappa} = 1 .$$

The evaluation of (4-32) is very similar to that of (4-23), and has been done by MacMillan. Eshelby gives the final form of φ_e as

$$\varphi_e = \frac{2\pi abc}{\ell^2} \left\{ \left[\ell^2 - \frac{x^2}{K^2} + \frac{y^2}{K^2} \right] F(\theta,K) + \left[\frac{x^2}{K^2} - \frac{y^2}{K^2K'^2} + \frac{z^2}{K'^2} \right] E(\theta,K) \right.$$

$$\left. + \frac{\ell}{K'^2} \left[\frac{CY^2}{AB} - \frac{BZ^2}{AC} \right] \right\} \qquad (4\text{-}33)$$

where

$$A = (a^2 + \varkappa)^{\frac{1}{2}}$$

$$B = (b^2 + \varkappa)^{\frac{1}{2}}$$

$$C = (c^2 + \varkappa)^{\frac{1}{2}}$$

$$\ell = (a^2 - c^2)^{\frac{1}{2}}$$

$$K'^2 = 1 - K^2 .$$

See the Appendix for the evaluation of the elliptic integrals and reductions of forms. Hence, by using equations (4-17), (4-30), and (4-33) the displacement field inside and outside of an ellipsoidal homogeneous inclusion embedded in an infinite matrix can be found explicitly.

All of the above discussion considers only a homogeneous inclusion of the same material as the matrix. Nevertheless, these results can be applied when the elastic constants of the matrix and inclusion differ. This can be done by use of an equivalent isotropic inclusion.

Suppose the included ellipsoid G' has elastic constants λ' and μ' while those of the matrix are λ and μ. Allow G' to undergo a stress-free strain $\underline{\underline{E}}^{T'}$. Apply surface tractions chosen so as to produce a uniform elastic strain $\underline{\underline{E}}^{c} - \underline{\underline{E}}^{c'}$ in the inclusion. Now, G' is in exactly in the same form as the original inclusion G considered above. If G' now has the same stresses in it as would be in G; then, G' may be replaced by G' without upsetting the continuity of displacement and surface tractions across B. The condition for this is

$$\underline{\underline{S}}^{T} = \lambda(\vartheta^{c} - \vartheta^{T})\underline{\underline{I}} + 2\mu(\underline{\underline{E}}^{c} - \underline{\underline{E}}^{T}) \qquad (4\text{-}34)$$

$$= \lambda'(\vartheta^{c} - \vartheta^{T'})\underline{\underline{I}} + 2\mu'(\underline{\underline{E}}^{c} - \underline{\underline{E}}^{T'}) \ .$$

When the values of λ', μ', and $\underline{\underline{E}}^{T'}$ are known (4-34) may be solved for $\underline{\underline{E}}^{T}$ and then the internal displacements, stresses, and strain may be found.

CHAPTER V

THE "SELF-CONSISTENT" METHOD

Another way to attempt to incorporate some of the effects of the interaction of inclusions in a matrix is to assume that the matrix material has the properties of the over-all bulk material. This technique is known as the "self-consistent" method or theory. This chapter will demonstrate the use of this theory, and show how a multiphase composite may be treated.

The previous chapters dealing with bulk properties have all dealt with two-phase composites. Suppose, however, that a heterogeneous material is composed of a coherent mixture of N isotropic elastic materials. The spatial distributions of the various constituents are assumed to be homogeneous and isotropic macroscopically; i.e., the material constants do not vary for different directions or locations in the media and across any arbitrary cross section the distribution of the inclusions is similar. Although in order to further describe such a heterogeneous media exactly some sort of deterministic or stochastic relation between the spatial distributions is needed; this will be omitted since only general composite media will be investigated and these may be specified qualitatively. The media is assumed to consist of irregular particles of the constituents which are distributed in a matrix in such a fashion that they are at arbitrary distances from one another. Moreover, the particles are assumed to be somewhat ellipsoidal so as to exclude flat plate-like shapes. This assumption is made because Eshelby (12) pointed out that the relationships avilable from the theory of elasticity are generally too complicated to be of any value, but these relationships may be solved for shapes such as ellipsoids. This was the context of the previous chapter. Budiansky (19) points out that the particle size need not be uniform, but that the size distribution must, macroscopically, be homogeneous; i.e., any arbitrary volume elements consisting of a cross sectional area and unit thickness must contain particles of different sizes in the same ratios. Thus, for an N-phase composite of volume V with volume V_i of the ith phase, the volume concentration $f_i = V_i/V$ is equivalent to the probability of finding that any given particle is composed of the i^{th} material. Budiansky also notes that for the limiting case of very small concentrations $f_1, f_2, \ldots, f_{N-1}$ which represent the first N-1 phases will tend to appear as isolated inclusions in a matrix consisting of the N^{th} phase.

Analysis

To determine the effective shear modulus μ^*, apply a uniform pure shear $\underline{\underline{S}}_\tau = \underline{\underline{S}}^o_\tau$ to the surface of a cube of the composite material. The corresponding shear strain $\underline{\underline{E}}_\gamma$ is not uniform throughout the cube. However, in Chapter III the value of the stress and strain fields was shown to equal the average value of these fields over the volume of the specimen. Hence, $\overline{\underline{\underline{E}}}_\gamma$ is the average value of the strain while $\overline{\underline{\underline{S}}}_\tau = \underline{\underline{S}}^o_\tau$. The shear modulus is defined as

$$\mu^* = \underline{\underline{S}}^o_\tau : \overline{\underline{\underline{E}}}^{-1}_\gamma \; . \tag{5-1}$$

The total strain energy is given by

$$U = \frac{1}{2} \int_D \underline{\underline{S}}^{o}_{\tau} : \underline{\underline{E}}_{\gamma}\, dv \tag{5-2}$$

$$= \frac{1}{2}(\underline{\underline{S}}^{o}_{\tau} : \overline{\underline{\underline{E}}}_{\gamma}) V$$

$$= \frac{\underline{\underline{S}}^{o}_{\tau} : \underline{\underline{S}}^{o}_{\tau}}{2\mu^*} V \ .$$

Moreover, in terms of the individual shear moduli μ_i $(i = 1,2,\ldots,N)$ of the various phases

$$U = \frac{1}{2} \int_D \frac{\underline{\underline{S}}^{o}_{\tau} : \underline{\underline{S}}^{o}_{\tau}}{\mu_N}\, dv + \frac{1}{2} \int_D \underline{\underline{S}}^{o}_{\tau} : \left(\underline{\underline{E}}_{\gamma} - \frac{\underline{\underline{S}}_{\tau}}{\mu_N}\right) dv \tag{5-3}$$

$$= \frac{\underline{\underline{S}}^{o}_{\tau} : \underline{\underline{S}}^{o}_{\tau}}{2\mu_N} V + \frac{\underline{\underline{S}}^{o}_{\tau}}{2} \sum_{i=1}^{N-1} \left(1 - \frac{\mu_i}{\mu_N}\right) \int_D \underline{\underline{E}}_{\gamma}\, dv$$

$$= \frac{\underline{\underline{S}}^{o}_{\tau} : \underline{\underline{S}}^{o}_{\tau}}{2} V \left[\frac{1}{\mu_N} + \sum_{i=1}^{N-1} f_i \left(1 - \frac{\mu_i}{\mu_N}\right)\left(\frac{\overline{\underline{\underline{E}}}_{\gamma_i}}{\underline{\underline{S}}^{o}_{\tau}}\right)\right]$$

where

$$\overline{\underline{\underline{E}}}_{\gamma_i} = \frac{1}{V_i} \int_{D_i} \underline{\underline{E}}_{\gamma}\, dv$$

is the average value of $\underline{\underline{E}}$ in the i^{th} phase. A comparison of (5-2) and (5-3) reveals that

$$\frac{1}{\mu^*} = \frac{1}{\mu_N} + \sum_{i=1}^{N-1} f_i \left(1 - \frac{\mu_i}{\mu_N}\right)\left(\frac{\overline{\underline{\underline{E}}}_{\gamma_i}}{\underline{\underline{S}}^{o}_{\tau}}\right) \ . \tag{5-4}$$

This relation is exact and consistent with the results of Chapters 2 and 3.

Now, to incorporate the self-consistent theory, approximate $\bar{\underline{\underline{E}}}_{\gamma_i}$ by the shear strain that would occur in an isolated spherical inclusion of the i^{th} material embedded in an infinite isotropic elastic matrix subjected to pure shear $\underline{\underline{S}}_\tau = \underline{\underline{S}}_\tau^o$ at infinity, and having the as-yet-unknown elastic properties of the composite material. The exact solution for this type of inclusion problem was obtained by Eshelby (12) as a special case of the ellipsoid. The actual shear strain in the inclusion was found to be uniform of magnitude

$$\underline{\underline{S}}_0 = \frac{\underline{\underline{S}}_\tau^o}{\mu^* + \beta^*(\mu_i - \mu^*)} \tag{5-5}$$

where

$$\beta^* = \frac{2(4-5\nu^*)}{15(1-\nu^*)}$$

and ν^* is the Poisson's ratio of the composite material. Now, using (5-5) as an approximation of $\bar{\underline{\underline{E}}}_{\gamma_i}$ and substituting into (5-4) yields

$$\frac{1}{\mu^*} = \frac{1}{\mu_N} + \sum_{i=1}^{N-1} \left(1 - \frac{\mu_i}{\mu_N}\right) \frac{f_i}{\mu^* + \beta^*(\mu_i - \mu^*)} \ . \tag{5-6}$$

A similar procedure can be followed in solving for the effective bulk modulus $\varkappa^*$. Let $\bar{\theta}$ be the average volumetric contraction due to the uniform hydrostatic pressure p^o acting on the surface B of the specimen. Then,

$$\varkappa^* = \frac{p^o}{\bar{\theta}} . \tag{5-7}$$

Using Eshelby's results again leads to

$$\frac{1}{\varkappa^*} = \frac{1}{\varkappa_N} + \sum_{i=1}^{N-1} \left(1 - \frac{\varkappa_i}{\varkappa_N}\right) \frac{f_i}{\varkappa^* + \alpha^*(\varkappa_i - \varkappa^*)} \tag{5-8}$$

where

$$\alpha^* = \frac{1+\nu^*}{3(1-\nu^*)} .$$

Recall the standard relation

$$\nu^* = \frac{3\varkappa^* - 2\mu^*}{6\varkappa^* + 2\mu^*} \tag{5-9}$$

Equations (5-6), (5-8), and (5-9) provide three equations in three unknowns $\varkappa^*$, μ^*, and ν^*; and can be solved for these unknowns. The self-consistent or "smearing-out" technique was first used with crystalline materials. It has been applied to mechanical properties of aelotropic materials by several authors. In fact Hill (18) has used this method to extend the results of his work which was discussed in Chapter III.

CHAPTER VI

CONCLUSIONS

Although this work has been an attempt give a survey of the methods used to analytically investigate anisotropic media, it is by no means complete. Only methods of predicting mechanical properties were discussed; whereas, there exist a vast amount of literature on the problem of predicting other properties such as dielectric and magnetic behavior, and conductivity and diffusivity of the media. The present work has dealt only with static approaches to the problem. There have been a few attempts to investigate the dynamical behavior of composites. Still, even with all of the readily available results on the subject, the entire field is in evolutionary stage.

There remains a great deal of work to be done in the study of aelotropic media. The dynamic view point has only been touched on slightly. A study of wave propagation should be made along with a study vibration characteristics of composite. Even in the much used static approaches there is a lot of work to be done. More complicated mechanical behavior needs to be investigated. Most of the work up to now has been confined to idealized cases; i.e., macroscopic homogeneity and isotropy of the media, small volume concentrations of inclusions in order to negate the perturbation effects, and depends upon volume fractions and physical constants of the constituents. Nevertheless, a general anisotropic media can only be defined stochastically due to the random distribution of the particles. Hence, the problem needs to be attacked from a statistical view point. In a similar manner, the existing theories almost exclusively deal with statistically homogeneous stress and strain fields and then generally deals only with the averages of these fields. Thus, the areas of statistically non-homogeneous fields and of microscopic internal field analysis are wide open for development.

There are a number of different classifications of anisotropic media; i.e.

viscous-viscous

viscous-elastic

viscous-plastic

elastic-elastic

elastic-plastic

plastic-plastic

where the left hand sides refers to the type of matrix media while the right hand side refers to the type of particle media. All of the present work deals with elastic-elastic media. Although much has been done in the analysis of the first two and fourth types of composites listed, investigations into the behavior of the other three types are not to be found in the literature except crude approximations.

The effects of other factors on the behavior of composite materials needs to be further investigated. For example, most analysis deal only with spherical or elliptical particles of uniform small size. The effects of various particle shapes and non-uniform sizes needs to be studied, perhaps only experimentally if analytical methods prove to be somewhat impossible. It is the author's belief that certain shapes of particles such as long thin fibers might be studied to see if they can be used to give the over-all media directional properties. By this is meant that by arranging the inclusions in an uni-directional manner the resultant effect would be, say, to stiffen the composite in the direction of the axis of the particles' orientation but not to any great extent in the perpendicular directions. This problem needs to be analytically investigated. Also, the dependence of the elastic moduli of composite materials on temperature and temperature gradients is of interest.

Finally, the need for extensive experimental work to check the results already available in the literature and to provide directions for further investigations must be emphasized. It is very possible that more experimental results would greatly aid in the understanding of some of the, analytically, complicating factors such as perturbation effects of the particles and effects of different shapes of inclusions.

Actually, more works of the present type where some of the ideas found in the literature are put into a common notation and related to another would be of great use, particularly to persons just getting interested in the field of composites.

APPENDIX A

DERIVATION OF THE NEWTONIAN POTENTIAL OF AN ELLIPSOID

For an interior point, φ is given by MacMillan (22) as

$$\varphi_i = \pi abc \int_0^\infty \left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{c^2+s}\right) \frac{ds}{[(a^2+s)(b^2+s)(c^2+s)]^{\frac{1}{2}}} . \qquad \text{(A-1)}$$

This integral may be evaluated in terms of four elliptic integrals. Let

$$[(a^2+s)(b^2+s)(c^2+s)]^{\frac{1}{2}} = \alpha(s)$$

$$\frac{a^2-c^2}{a^2} = \beta^2$$

$$A_1 = \int_0^\infty \frac{dS}{\alpha(s)} , \qquad \text{let } \omega = \sqrt{\frac{a^2-c^2}{a^2+s}}$$

$$a^2+s = \frac{a^2-c^2}{\omega^2}$$

$$ds = -2\,\frac{a^2-c^2}{\omega^3}\,d\omega$$

$$\therefore \qquad A_1 = \int_\beta^0 \frac{\left(\frac{-2}{\omega^3}\right)(a^2-c^2)\,d\omega}{\left[\left(\frac{a^2-c^2}{\omega^2}\right)\left(\frac{a^2-c^2}{\omega^2} + b^2-a^2\right)\left(\frac{a^2-c^2}{\omega^2} + c^2-a^2\right)\right]^{\frac{1}{2}}} \qquad \text{(A-2)}$$

$$= \int_0^\beta \frac{\left(\frac{2}{\omega^3}\right)(a^2-c^2)\,d\omega}{\left(\frac{1}{\omega^3}\right)[(a^2-c^2)(a^2-c^2+b^2\omega^2-a^2\omega^2)(a^2-c^2+c^2\omega^2-a^2\omega^2)]^{\frac{1}{2}}}$$

$$= \int_0^{\beta} \frac{2(a^2-c^2)d\omega}{\{[(a^2-c^2)^2 + \omega^2(b^2-a^2)(a^2-c^2)](1-\omega^2)(a^2-c^2)\}^{\frac{1}{2}}}$$

$$= \int_0^{\beta} \frac{2d\omega}{[(1-\omega^2)(1-K^2\omega^2)](a^2-c^2)^{\frac{1}{2}}}$$

where

$$K^2 = \frac{a^2-b^2}{a^2-c^2} .$$

Let

$$\omega = \sin\theta , \quad d\omega = \cos\theta \, d\theta$$

$$A_1 = \int_0^{\omega_s} \frac{2\cos\theta \, d\theta}{(a^2-c^2)^{\frac{1}{2}}[\cos^2\theta)(1-K^2\sin^2\theta)]^{\frac{1}{2}}} \; ; \quad \omega_s = \sin^{-1}\left[\frac{a^2-c^2}{a^2}\right] = \theta$$

$$= \int_0^{\omega_s} \frac{2}{(a^2-c^2)^{\frac{1}{2}}} \frac{d\theta}{[1-K^2\sin^2\theta]^{\frac{1}{2}}}$$

$$A_1 = \frac{2}{(a^2-c^2)^{\frac{1}{2}}} F(\theta, K) \tag{A-3}$$

where $F(\theta, K)$ is Legendre's elliptic integral of the first kind. Define

$$v = \frac{(a^2-c^2)^{\frac{1}{2}}}{2} A_1 = F(\theta, K)$$

$$A_2 = \int_0^{\infty} \frac{ds}{(a^2+s)\alpha(s)} = \int_0^{\beta} \frac{2d\omega}{\frac{a^2-c^2}{\omega^2}[(1-\omega^2)(1-K^2\omega^2)]^{\frac{1}{2}}(a^2-c^2)^{\frac{1}{2}}} \tag{A-4}$$

$$= \int_0^{\omega_s} \frac{2}{(a^2-c^2)^{3/2}} \frac{\sin^2\theta d\theta}{[1-K^2\sin^2\theta]^{\frac{1}{2}}} .$$

Let

$$\text{snv} = \sin\,\omega_s = \sqrt{\frac{a^2-c^2}{a^2}} .$$

Observe

$$\frac{ds}{\alpha(s)} = \frac{-2}{(a^2-c^2)^{\frac{1}{2}}}\,dv$$

$$dv = \frac{d\theta}{[1-K^2\sin^2\theta]^{\frac{1}{2}}}$$

$$\therefore A_2 = \frac{2}{(a^2-c^2)^{3/2}} \int_0^{v} \text{sn}^2 v\;dv = \frac{2}{(a^2-c^2)^{3/2}} \frac{1}{K^2} [F(\theta,K) - E(\theta,K)] \qquad \text{(A-5)}$$

where $E(\theta,K)$ is Legendre's elliptic integral of the second kind.

$$A_3 = \int_0^{\infty} \frac{ds}{(b^2+s)\alpha(s)} ; \qquad (b^2+s) = \frac{a^2-c^2}{\omega^2} - a^2 + b^2$$

$$A_3 = \int_0^{\beta} \frac{2}{(a^2-c^2)^{\frac{1}{2}}} \frac{d\omega}{[(1-\omega^2)(1-K^2\omega^2)]^{\frac{1}{2}}\left[\frac{a^2-c^2}{\omega^2} - a^2 + b^2\right]} \qquad \text{(A-6)}$$

$$= \frac{2}{(a^2-c^2)^{\frac{1}{2}}} \int_0^{\beta} \frac{\omega^2 d\omega}{[(1-\omega^2)(1-K^2\omega^2)]^{\frac{1}{2}}[a^2(1-\omega^2)-c^2+b^2\omega^2]}$$

$$= \frac{2}{(a^2-c^2)^{\frac{1}{2}}} \int_0^{\omega_s} \frac{\sin^2\theta d\theta}{[1-K^2\sin^2\theta]^{\frac{1}{2}}[(a^2-c^2)+\sin^2\theta(b^2-a^2)]}$$

$$= \frac{2}{(a^2-c^2)^{3\ 2}} \int_0^{\omega_s} \frac{\sin^2\theta d\theta}{[1-K^2\sin^2\theta]^{\frac{1}{2}}[1-K^2\sin^2\theta]}$$

$$[1-K^2\sin^2\theta] = 1 - \frac{a^2-b^2}{a^2} \quad \frac{b^2}{a^2} .$$

Let

$$\mathrm{dnv} = \frac{b}{a}$$

$$\therefore \qquad A_3 = \frac{2}{(a^2-c^2)^{3/2}} \int_0^{v} \frac{\mathrm{sn}^2 v}{\mathrm{dn}^2 v}\, dv$$

$$= \left[\frac{2}{(a^2-c^2)^{3\ 2}}\right]\left[\frac{E(\theta,K)}{K^2(1-K^2)} - \frac{1}{1-K^2}\ \frac{\mathrm{snv\ cnv}}{\mathrm{dnv}} - \frac{F(\theta,K)}{K^2}\right] \qquad \text{(A-7)}$$

where $\mathrm{cnv} = \frac{c}{a}$

$$A_4 = \int_0^{\infty} \frac{ds}{(c^2+s)\alpha(s)} \ ; \ c^2+s = \frac{a^2-c^2}{\omega^2} - a^2 + c^2$$

$$A_4 = \int_0^{\beta} \frac{2}{(a^2-c^2)^{\frac{1}{2}}} \frac{d\omega}{[(1-\omega^2)(1-K^2\omega^2)]^{\frac{1}{2}}\left[\frac{a^2-c^2}{\omega^2} - a^2 + c^2\right]} \qquad \text{(A-8)}$$

$$= \frac{2}{(a^2-c^2)^{\frac{1}{2}}} \int_0^{\beta} \frac{\omega^2 d\omega}{[(1-\omega^2)(1-K^2\omega^2)]^{\frac{1}{2}}[(a^2-c^2)(1-\omega^2)]}$$

$$= \frac{2}{(a^2-c^2)^{3/2}} \int_0^{\omega_s} \frac{\sin^2\theta d\theta}{[1-K^2\sin^2\theta]^{\frac{1}{2}}(1-\sin^2\theta)}$$

$$(1-\sin^2\theta) = 1 - \frac{a^2-c^2}{a^2} = \frac{c^2}{a^2} = \mathrm{cn}^2 v$$

$$\therefore \qquad A_4 = \frac{2}{(a^2-c^2)^{3/2}} \int_0^v \frac{sn^2v}{cn^2v}\,dv$$

$$= \frac{2}{(a^2-c^2)^{3/2}} \left[\frac{1}{1-K^2}\left(\frac{snv\,dnv}{cnv} - E(\theta,K)\right)\right] . \qquad (A-9)$$

Hence, using (A-3), (A-5), (A-7), and (A-9) one may rewrite (A-1) as

$$\varphi_i = \frac{2\pi abc}{(a^2-c^2)^{\frac{1}{2}}} \left\{F(\theta,K) - \frac{x^2}{(a^2-b^2)}\,[F(\theta,K) - E(\theta,K)]\right.$$

$$- \frac{E(\theta,K)y^2}{(a^2-b^2)(1-K^2)} + \frac{F(\theta,K)y^2}{(a^2-b^2)} + \frac{y^2}{(1-K^2)}\,\frac{snv\;cnv}{dnv}$$

$$\left. + \frac{E(\theta,K)z^2}{(1-K^2)(a^2-c^2)} - \frac{z^2}{(a^2-c^2)}\,\frac{snv\;cnv}{cnv}\right\} . \qquad (A-10)$$

Making the proper substitutions for snv , cnv , and dnv and rearranging (A-10) yields

$$\varphi_i = \frac{2\pi abc}{(a^2-c^2)^{\frac{1}{2}}} \left\{\left[1 - \frac{x^2}{a^2-b^2} + \frac{y^2}{a^2-c^2}\right] F(\theta,K)\right.$$

$$+ \left[\frac{x^2}{a^2-b^2} - \frac{(a^2-c^2)y^2}{(a^2-b^2)(b^2-c^2)} + \frac{z^2}{b^2-c^2}\right] E(\theta,K)$$

$$\left. + \left[\left(\frac{c^2y^2}{b^2-c^2} - \frac{b^2z^2}{b^2-c^2}\right)\left(\frac{\sqrt{a^2-b^2}}{abc}\right)\right]\right\} . \qquad (A-11)$$

Eshelby has written φ_i in the following form

$$\varphi_i = \tfrac{1}{2}(a^2-x^2)I_a + \tfrac{1}{2}(b^2-y^2)I_b + \tfrac{1}{2}(c^2-z^2)I_c \qquad (A-12)$$

$$I_a = \frac{4\pi abc}{(a^2-b^2)(a^2-c^2)^{\frac{1}{2}}} \left[F(\theta,K) - E(\theta,K)\right]$$

$$I_b = 4\pi - I_a - I_c$$

$$I_c = \frac{4\pi abc}{(b^2-c^2)(a^2-c^2)^{\frac{1}{2}}} \left[\frac{b(a^2-c^2)^{\frac{1}{2}}}{ac} - E(\theta,K)\right] \quad .$$

Using the given expressions for I_a, I_b, and I_c, (A-12) is found to be equivalent to (A-11).

The expression for φ for an exterior point can be given as

$$\varphi_e = \pi abc \int_{\varkappa}^{\infty} \left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{c^2+s}\right)\left(\frac{ds}{\alpha(s)}\right) \quad . \tag{A-13}$$

MacMillan shows how to evaluate this integral. The approach is very similar to that used for φ_i. As a matter of fact, the only difference is that $\varkappa$ is the lower limit of integration instead of zero; and, thus, the following changes occur

$$\mathrm{sn}\,v_{\varkappa} = \sqrt{\frac{a^2-c^2}{a^2+\varkappa}} \tag{A-14}$$

$$\mathrm{cn}\,v_{\varkappa} = \sqrt{\frac{c^2+\varkappa}{a^2+\varkappa}}$$

$$\mathrm{dn}\,v_{\varkappa} = \sqrt{\frac{b^2+\varkappa}{a^2+\varkappa}} \quad .$$

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